One of the problems representing spin is that in three-dimensional space we have three directions: x, y, and z. However, the spin state space works with complex-valued vector spaces. A two-dimensional complex vector space corresponds to a four-dimensional real vector space, which we can no longer easily visualize.

We express the possible spin states in the $|0\rangle/|1\rangle$ basis. This is the orientation of the spin along the z-axis.

Spin states along the z-axis:	Spin states along the x-axis:	Spin along the y-axis:
$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$	$|x+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $|x+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $|x-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ $|x-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$	$|y+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ $|y+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ $|y-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ $|y-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

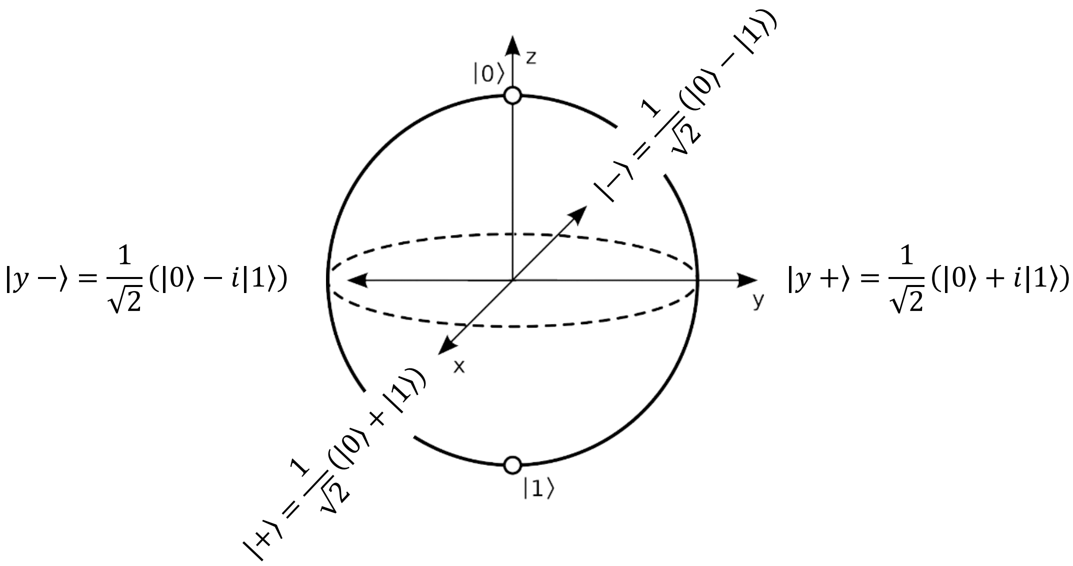
Note: $|x+\rangle$ usually written as $|+\rangle$, $|x-\rangle$ usually written as $|-\rangle$

Note: $|y+\rangle$ often written as $|i+\rangle$, $|y-\rangle$ often written as $|i-\rangle$

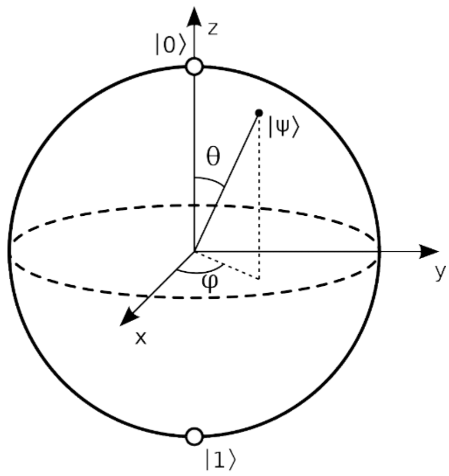
The orientation of spin is visualized by using the Bloch-Sphere.

A spin orientated spherical in the z-direction is written as $|0\rangle$ and corresponds to the complex-valued vector $\begin{pmatrix}1\\0\end{pmatrix}$. A spin orientated spherical in the $-z$-direction is written as $|1\rangle$ and corresponds to the complex-valued vector $\begin{pmatrix}0\\1\end{pmatrix}$. Please note that $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are orthogonal vectors, whereas the spherical direction z and $-z$ are not.

Using the $|0\rangle/|1\rangle$-basis we assign:



Please note that the spherical directions x and $-x$ as well as y and $-y$ are not orthogonal, the states $|+\rangle$, $|-\rangle$ and $|y+\rangle$, $|y-\rangle$ are orthogonal.



An arbitrary state $|\psi\rangle$ can be expressed in the $|0\rangle/|1\rangle$-basis by first rotating it by the angle θ with respect to the z-axis and then by the angle φ with respect to the x-axis.

We have to deal with the spherical/complex problematic again.

Rotating a spherical vector by 90° resp. by $\frac{\pi}{2}$ produces an orthogonal spherical vector.

Rotating a complex state by 180° resp. by π produces an orthogonal complex state.

We show this by an example, setting $\theta = \frac{\pi}{2}$ and $\varphi = 0$. Obviously, the state $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ will change to the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

In contrast, if we take the spherical vector $\begin{pmatrix}1\\0\end{pmatrix}$ and rotate it by $\frac{\pi}{2}$ it will change into the vector $-\begin{pmatrix}0\\1\end{pmatrix}$ and this is an orthogonal direction spherical.

Rotating is performed by using unitary matrices. An invertible complex valued square matrix U is unitary if its conjugate transpose is also its inverse:

$$U^\dagger U = UU^\dagger = I$$

Rotation preserves the length (norm) of a vector. In quantum mechanics the length of a vector corresponds to its probability amplitude, so a rotation doesn't change it.

Note: The rows and columns of a unitary matrix form an orthonormal basis of $\mathbb{C}^n$.

Rotation around one of the primary axes is performed by help of the Pauli matrices:

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

The Pauli matrices are unitary:

$$\sigma_x{}^\dagger = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \sigma_y{}^\dagger = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \sigma_z{}^\dagger = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

We check unitarity:

$$\sigma_x{}^\dagger \sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \{1\}$$

$$\sigma_y{}^\dagger \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \{2\}$$

$$\sigma_z{}^\dagger \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \{3\}$$

We apply rotation around the y-axis to the state vector $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$:

$$\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\i\end{pmatrix} = -i\begin{pmatrix}0\\1\end{pmatrix} \quad \{4\}$$

We call the factor $-i$ a global phase. A global phase has no effect to quantum mechanical measurements and can be ignored in most cases.

We rewrite $-i = e^{i\frac{3}{2}\pi}$ and get:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{i\frac{3}{2}\pi} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Obviously, the rotation angle is π resp. 180°. Rotating twice by use of the Pauli matrices restores the original state:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} e^{i\frac{3}{2}\pi} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\frac{3}{2}\pi} \cdot -i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{i\frac{6}{2}\pi} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \{5\}$$

Note: $e^{i\frac{6}{2}\pi} = e^{i2\pi} = 1$

The vector $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ should rotate to the vector $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. We check this:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -i\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \{6\}$$

Again, we use that $-i = e^{i\frac{3}{2}\pi}$ and get:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{i\frac{3}{2}\pi} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We rotate again:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} e^{i\frac{3}{2}\pi} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{i\frac{3}{2}\pi} \cdot i \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \quad \{7\}$$

$$e^{i\frac{3}{2}\pi} \cdot e^{i\frac{1}{2}\pi} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Besides working with explicit representation of vectors we can work on the conceptional level.

The calculations above, $\{1\}$ til $\{7\}$ give:

$$\{1\}\ {\sigma_x}^\dagger \sigma_x = Id\ ,\ \{2\}\ {\sigma_y}^\dagger \sigma_y = Id,\ \{3\}\ {\sigma_z}^\dagger \sigma_z = Id$$

$$\{4\}\ \sigma_y|0\rangle = |1\rangle$$

$$\{5\}\ \sigma_y|1\rangle = |0\rangle$$

$$\{6\}\ \sigma_y|+\rangle = |-\rangle$$

$$\{7\}\ \sigma_y|-\rangle = |+\rangle$$

So far, we can rotate the Bloch sphere around the axes x, y, z by π resp. 180°.

If we want to interchange two points on the Bloch sphere, we can build this rotation with a combination of a maximum of three rotations around the axes x, y, z.

In order to rotate the Bloch sphere round the $x/y/z$-axis by an angle of θ we apply the unitary matrix:

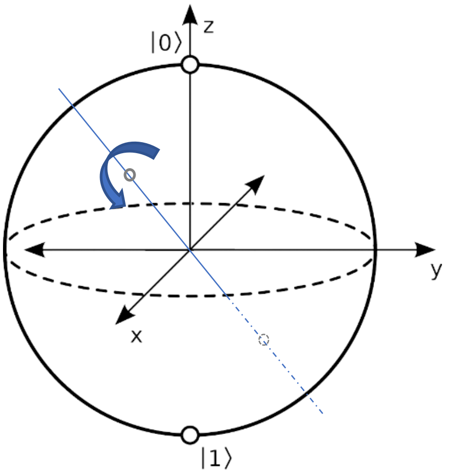
$$e^{-i\cdot\frac{\theta}{2}\cdot\sigma_x} = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) & -i\cdot sin\left(\frac{\theta}{2}\right) \\ -i\cdot sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$e^{-i\cdot\frac{\theta}{2}\cdot\sigma_y} = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) & -sin\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$e^{-i\cdot\frac{\theta}{2}\cdot\sigma_z} = \begin{pmatrix} e^{-i\cdot\frac{\theta}{2}} & 0 \\ 0 & e^{i\cdot\frac{\theta}{2}} \end{pmatrix} = e^{-i\cdot\frac{\theta}{2}}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\cdot\theta} \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\cdot\theta} \end{pmatrix}$$

Note: for the proof you may refer to
“exponentiation_of_pauli_matrices” on this website within the part of quantum mechanics or
https://www.researchgate.net/publication/335654130_Rotations_on_the_Bloch_Sphere.

The Hadamard matrix H rotates around the $(x+z)$-axis by π resp. 180°:



It flips x- and z-axis:

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$	$H\|0\rangle = \|+\rangle$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$H\|+\rangle = \|0\rangle$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$	$H\|1\rangle = \|-\rangle$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$H\|-\rangle = \|1\rangle$