This is a collection of mathematics needed to start.

Hope I can help you with learning quantum mechanics.

Contents

Formal Logic.

Formal logic or classical logic works with 1 and 0 ore true and false.

A statement: If it is raining, the street becomes wet.

In mathematics we use the name "proposition" for statement.

A proposition can be true or false. Mathematically propositions can be equations like $2 + 3 = 5$. An equation like $2 \cdot x + 3 = 7$ is not a proposition but a propositional expression that becomes true or false depending on what you insert for the variable x.

If you concatenate propositions with "and" resp. the "or", true or false for the composite proposition follow rules.

Let A and B be propositions, then the truth values (A "or" B) are:

A	B	$A \text{ or } B$
0	0	0
0	1	1
1	0	1
1	1	1

The truth values for (A "and" B):

A	B	$A \text{ and } B$
0	0	0
0	1	0
1	0	0
1	1	1

There is a special logical operator ¬, the "not" that simply switches the truth value to its opposite:

A	$\neg A$
0	1
1	0

In quantum mechanics often used is the orientation of a spin in space. This spin can be "up" or "down".

We need a way to describe these orientations.

A simple variable with e.g., 1 for "spin up" and 0 for "spin down" is not sufficient, we use two variables for the orientation, one representing "up", the other "down". As they are logically connected to each other, we write them as $\binom{a}{b}$:

$\binom{1}{0}$ defines the state with the spin-vector "up",

$\binom{0}{1}$ the state with the spin-vector "down".

This fits with our formal logic. The proposition "the spin is up **or** down" is true for both combinations:

A	B	$A \text{ or } B$
0	1	1
1	0	1

The proposition “the spin is up **and** down” is false for both combinations:

A	B	$A\ and\ B$
0	1	0
1	0	0

We need a method to show that the positions “up” and “down” are mutually exclusive. We adopt the dot product: $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$ and apply it:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot 0 + 0 \cdot 1 = 0$$

Unfortunately, things are a little bit more complicated in quantum mechanics. The result of measuring the spin gives either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The spin itself can take all combinations in between: $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with α and β being fractions between 0 and 1.

We standardize the notation for the spin positions: $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ merge to $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

We check the proposition “the spin is $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$” and “the spin is $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$”, this should give +1 too:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \cdot \alpha + \beta \cdot \beta = \alpha^2 + \beta^2$$

As the result should be 1, we get a normalization condition: $\alpha^2 + \beta^2 = 1$.

Constantly writing terms like $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is cumbersome, therefore Dirac invented the symbol $|u\rangle$ for $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|d\rangle$ for $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

He named this notation “ket”. Using kets instead of $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we can write more easily $\alpha|u\rangle$ and $\beta|d\rangle$.

Throughout quantum mechanics there will be a constant change between all possible notations, because for some problems special notations fit best.

Conclusion:

The state of a spin can be described by:

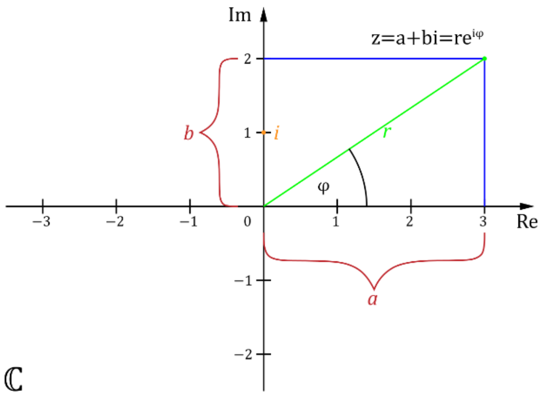
- verbal, spin is up, or spin is down,
- by a symbol $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ with polar opposites $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
- by the expression $|u\rangle$ for “up” and $|d\rangle$ for “down”,
- by the expression $\alpha|u\rangle + \beta|d\rangle$ for a generalized spin
- the normalization condition $\alpha^2 + \beta^2 = 1$.

Note: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ etc. can be used as vectors in the real R^2-space. The spin has an orientation in real space, but the internal relations between “up” and “down” have not.

Complex numbers

A complex number consists of a real part and an imaginary part. We write it as $z = a + ib$ with $a, b \in \mathbb{R}$. The imaginary unit i has the property $i \cdot i = -1$ or $i^2 = -1$.

We can represent complex numbers by a plane with the horizontal real axis and the vertical imaginary axis. This is called the cartesian mode.



... graphic courtesy of Wikipedia ...

A second way of representation describes a complex number by the angle it has with the real axis and its length resp. the absolute value. This is called the gaussian mode. In this mode we write a complex number as $r \cdot e^{i\varphi}$.

We can switch from one representation to the other:

Given $z = a + ib$: $|z|\ or\ r = \sqrt{a^2 + b^2}$ $\varphi = \arccos\left(\frac{a}{r}\right)$ if $b \geq 0$

resp. $\varphi = -\arccos\left(\frac{a}{r}\right)$ if $b < 0$.

Given $z = re^{i\varphi}$: $a = r \cdot \cos(\varphi)$ $b = r \cdot \sin(\varphi)$

or $z = r \cdot (\cos(\varphi) + i \cdot \sin(\varphi))$

Every complex number z has a complex conjugate number, marked as $\bar{z}$ or z^*. The complex conjugate switches the imaginary part to the opposite sign. $z = a + ib$ changes to $\bar{z} = a - ib$ and vice versa.

With that we get new formulas:

$$|z| = \sqrt{z\bar{z}}$$

Note: $\sqrt{z\bar{z}}$ is a positive real number.

$$re(z)\ or\ a = \frac{z + \bar{z}}{2}$$

$$im(z)\ or\ b = \frac{z - \bar{z}}{2}$$

Additions and subtraction of complex numbers are best performed with the cartesian representation.

$$(a + ib) + (c + id) = ac + ibd$$

Multiplication and division are best performed with the gaussian representation.

$$r_1 e^{i\varphi} \cdot r_2 e^{i\theta} = r_1 r_2 e^{i(\varphi + \theta)}$$

Matrices with real coefficients

The logical not-operator acting on the state of a spin reverses it. With our choice of writing the state of a spin as $\begin{pmatrix}1\\0\end{pmatrix}$ we need a matrix for the reverse operation. The matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ performs this:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\cdot1+1\cdot0\\1\cdot1+0\cdot0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\cdot0+1\cdot1\\1\cdot0+0\cdot1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

In quantum mechanics matrices often are referred to as “operators”. If we refer to the matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ as “negation operator” *neg* and using the Dirac ket, we write the calculations above as

$$neg|u\rangle=|d\rangle$$

$$neg|d\rangle=|u\rangle$$

A matrix acting on a vector produces a new vector like in the examples above.

A matrix acting on a matrix produces a new matrix:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0\cdot0+1\cdot1&0\cdot1+1\cdot0\\1\cdot0+0\cdot1&1\cdot1+0\cdot0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is called the identity matrix I_d.

We call a matrix A the inverse to matrix B, if $A\cdot B=I_d$ and in this case write $B=A^{-1}$. Sometimes it happens that a matrix is its own inverse like in the upper example the matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

There is a scheme to produce the inverse matrix, look for “Gauss-Jordan-Algorithm” on the internet.

If we sum up the elements on the diagonal of a matrix, we get the trace of the matrix. This works only with quadratic matrices.

A matrix having elements $\neq 0$ only at the diagonal is called a diagonal matrix, the identity matrix being an example of.

A matrix A can be transposed to A^t meaning that it is mirrored at the diagonal.

$$A:=\begin{pmatrix}a&b\\c&d\end{pmatrix}\to A^t=\begin{pmatrix}a&c\\b&d\end{pmatrix}$$

$$A:=\begin{pmatrix}a&b&c\\d&e&f\end{pmatrix}\to A^t=\begin{pmatrix}a&d\\b&d\\c&f\end{pmatrix}$$

A vector is a special form of a matrix and can be written in two ways:

as m× 1-matrix, a column vector $\vec{x}=\begin{pmatrix}x_1\\\vdots\\x_m\end{pmatrix}$

as $1\times n$-matrix, a row vector $\vec{x}=(x_1\ \dots\ x_n)$

To work with matrices the number of lines of the first matrix must be equal to the number of columns of the second matrix. For quadratic matrices this means they must have the same size.

An example for rectangular matrices:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} k & l \\ m & n \\ o & p \end{pmatrix} = \begin{pmatrix} a \cdot k + b \cdot m + c \cdot o & a \cdot l + b \cdot n + c \cdot p \\ d \cdot k + e \cdot m + f \cdot o & d \cdot l + e \cdot n + f \cdot p \end{pmatrix} = \begin{pmatrix} u & v \\ w & x \end{pmatrix}$$

If matrices act on matrices the commutative law isn't always valid. Let A and B matrices with $A \neq B$, then normally $A \cdot B \neq B \cdot A$. Example:

$$A \cdot B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 5 + 1 \cdot 2 & 3 \cdot 2 + 1 \cdot 1 \\ 3 \cdot 2 + 1 \cdot 1 & 2 \cdot 2 + 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} 17 & 7 \\ 7 & 8 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 \cdot 3 + 2 \cdot 2 & 5 \cdot 1 + 1 \cdot 4 \\ 2 \cdot 3 + 1 \cdot 2 & 2 \cdot 1 + 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 19 & 9 \\ 8 & 6 \end{pmatrix}$$

This plays an important role in quantum mechanics and led to the definition of a commutator. The commutator of the matrices A and B:

$$[A, B] = A \cdot B - B \cdot A$$

For our example we get:

$$\left[\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \right] = \begin{pmatrix} 17 & 7 \\ 7 & 8 \end{pmatrix} - \begin{pmatrix} 19 & 9 \\ 8 & 6 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -1 & 2 \end{pmatrix}$$

If the commutator of two matrices is zero, the matrices commute.

A real number r has an absolute value written as $|r|$.

A complex number z has an absolute value written as $|z|$.

A matrix has something like an absolute value that is called the *determinant* of the matrix. For a 2×2-matrix the determinant is easy to calculate:

$$\left| \begin{pmatrix} 3 & 5 \\ 7 & 11 \end{pmatrix} \right| = (3 \cdot 11 - 5 \cdot 7) = -2$$

For a 3×3-matrix the calculation scheme is more tedious:

$$\left| \begin{pmatrix} 1 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{pmatrix} \right| = 1 \cdot \left| \begin{pmatrix} 11 & 13 \\ 19 & 23 \end{pmatrix} \right| - 3 \cdot \left| \begin{pmatrix} 7 & 13 \\ 17 & 23 \end{pmatrix} \right| + 5 \cdot \left| \begin{pmatrix} 7 & 11 \\ 17 & 19 \end{pmatrix} \right| = -84$$

If a matrix A acts on a vector, the result normally is another vector. There are rare cases a matrix acting on a vector produces the same vector or a multiple of it (the identity matrix e.g. always replicates the vector applied to it). Such vectors are called "eigenvector" to the matrix A. The multitude is called "eigenvalue" of the eigenvector.

To find eigenvectors we first must check the determinant of the matrix, it must be $\neq 0$:

$$\det(A) \coloneqq |A| \neq 0$$

Example:

$$A \coloneqq \begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}$$

$$|A| = \left| \begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix} \right| = -9$$

We calculate the characteristic polynomial to get the eigenvalues:

$$\det(A - \lambda I) = 0$$

$$|A - \lambda I| = \left|\begin{pmatrix} 1-\lambda & 8 \\ 2 & 7-\lambda \end{pmatrix}\right| = 0$$

$$(1-\lambda)(7-\lambda) - 16 = 0$$

$$\lambda^2 - 8\lambda - 9 = 0$$

Solution:

$$\lambda_{1/2} = \frac{8 \mp 10}{2}$$

$$\lambda_1 = 9, \lambda_2 = -1$$

There must exist vectors that fulfill:

$$\begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

We use the first eigenvalue $\lambda_1 = 9$:

$$I: a + 8b = 9a$$

$$II: 2a + 7b = 9b$$

From I: we get

$$b = a$$

From II: we get

$$b = a$$

Our first eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and every multiple of it.

We use the second eigenvalue $\lambda_2 = -1$:

$$I: a + 8b = -a$$

$$II: 2a + 7b = -b$$

From I: we get

$$b = -\frac{a}{4}$$

From II: we get

$$a = a$$

Our second eigenvector is $\begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix}$ and every multiple of it.

We check our solution for eigenvector 1:

$$A \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix} = 9 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We check our solution for eigenvector 2:

$$A \cdot \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{1}{4} \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix}$$

In quantum mechanics these equations often are written with kets:

$$operator\ A := \begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} := |\lambda_1\rangle\ and\ \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix} := |\lambda_2\rangle$$

$$A|\lambda_1\rangle = \lambda_1|\lambda_1\rangle = 9|\lambda_1\rangle$$

$$A|\lambda_2\rangle = \lambda_2|\lambda_2\rangle = -1|\lambda_2\rangle$$

Eigenvectors are linearly independent to each other.

Note: eigenvectors are not necessarily orthogonal to each other.

Standard basis of a cartesian vector space are vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ but every other couple of linear independent vectors can serve too. This way we can use eigenvectors as a new basis of the space they were born.

The dot-product:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 5 = 13$$

If $\vec{a} \cdot \vec{b} = 0$ then the two vectors are orthogonal to each other.

You transpose $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to $(1\ 2)$ and multiplicate row vector with column vector: $(1\ 2) \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 13$.

This can be done the other way around and gives a surprising result: $\begin{pmatrix} 3 \\ 5 \end{pmatrix} \cdot (1\ 2) = \begin{pmatrix} 3 & 6 \\ 5 & 10 \end{pmatrix}$.

Matrices with complex coefficients

Quantum mechanics deals with the contrast $real\ number \leftrightarrow complex\ number$.

You can complex conjugate a matrix:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \rightarrow A^* = \begin{pmatrix} a^* & b^* & c^* \\ d^* & e^* & f^* \\ g^* & h^* & k^* \end{pmatrix}$$

If a matrix and its complex conjugated are identic then the matrix must have real coefficients only.

$$A = A^* \rightarrow a_{ij} \in \mathbb{R}$$

A matrix U is called unitary if: you transpose it $U \rightarrow U^t$, build the complex conjugate $U^t \rightarrow U^{t*}$, multiply the original matrix and get the identity:

$$UU^{t*} = I = U^{t*}U$$

In physics this is called Hermitian adjoint: $U^\dagger U = UU^\dagger = I$. Please note the symbol †.

Example: $U \coloneqq \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

$$UU^{\dagger} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix} =$$

$$\frac{1}{4}\begin{pmatrix} (1+i)(1-i)+(1-i)(1+i) & (1+i)(1+i)+(1-i)(1-i) \\ (1-i)(1-i)+(1+i)(1+i) & (1-i)(1+i)+(1+i)(1-i) \end{pmatrix} =$$

$$\frac{1}{4}\begin{pmatrix} (1+1)+(1+1) & (1+2i-1)+(1-2i-1) \\ (1-2i-1)+(1+2i-1) & (1+1)+(1+1) \end{pmatrix} =$$

$$\frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For row vectors and column vectors the following rule applies: if you switch between column and row vector you have to complex conjugate: the column vector $\bar{a}{:}= |a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ has the corresponding row vector $(a_1^* \; a_2^*) \coloneqq \langle a^*|$, the latter called bra $\langle a^*|$.

The combination $\langle a|a\rangle$ delivers a scalar product and is called the inner product. The combination $|a\rangle\langle a|$ is called the outer product and delivers a matrix or an operator.

This leads to the following behavior:

$$\langle a|b\rangle = \langle b^*|a^*\rangle = \langle b|a\rangle^*$$

Tensor products

We can combine two vector spaces by building the tensor product of both.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

In quantum mechanics we need the tensor product of two spin-states.

We remember the symbols $|u\rangle$ for $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|d\rangle$ for $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or in combined form: $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for $\alpha|u\rangle + \beta|d\rangle$.

Combining two spin states into a product space (the tensor product space):

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \beta\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

Especially important is the tensor product for the states up and down for each spin:

$$|u\rangle\otimes|u\rangle \coloneqq |uu\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|u\rangle \otimes |d\rangle \coloneqq |ud\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|d\rangle \otimes |u\rangle \coloneqq |du\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}1\\0\end{pmatrix}\\1\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|d\rangle \otimes |d\rangle \coloneqq |dd\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\begin{pmatrix}0\\1\end{pmatrix}\\1\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The vectors $|u\rangle$ and $|d\rangle$ form a basis for the spin state. The tensor product of these two basis vectors forms a basis for the tensor state of two spins. The expression $|uu\rangle$ has always the meaning of $|u\rangle$ for the first spin and $|u\rangle$ for the second spin etc.

Calculus

The derivative of a function is defined as:

$$f'(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

In physics the derivative in respect to time $\frac{df}{dt}$ is often written as $\dot{f}$.

If f is a function of more than one variable we can partially derive it. In this case we chose one variable and regard all other being constant.

$$f(x, y, z) \coloneqq x + 2xy + z^3$$

$$\frac{\partial f}{\partial x} = 1 + 2y, \qquad \frac{\partial f}{\partial y} = 2x, \qquad \frac{\partial f}{\partial z} = 3z^2$$

$$\frac{\partial^2 f}{\partial x^2} = 0, \qquad \frac{\partial^2 f}{\partial y^2} = 0, \qquad \frac{\partial^2 f}{\partial z^2} = 6z$$

Integration by parts:

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

Note: if $u(x)$ and (x) are normalized and the integration performed from $-\infty$ to ∞ this becomes:

$$\int_{-\infty}^{\infty} u(x)v'(x)dx = -\int_{-\infty}^{\infty} u'(x)v(x)dx$$

The expression $[u(x)v(x)]_{-\infty}^{\infty}$ vanishes because $u(x)$ and (x) becomes zero in infinity.

A differential equation connects a function and some of its derivatives, e.g.:

$$f'(x) = 2 \cdot f(x)$$

The exponential function $f(x) = e^{2x}$ is a solution to this differential equation, because:

$$f'(e^{2x}) = 2 \cdot e^{2x} = 2 \cdot f(x)$$

Besides the function $f(x) = e^{2x}$ every multiple $f(x) = ae^{2x}$ is solution to this differential equation:

$$f'(ae^{2x}) = af'(e^{2x}) = a \cdot 2 \cdot e^{2x} = 2 \cdot ae^{2x} = 2 \cdot f(x)$$

Mathematicians view to functions concentrates on what the function does with the variable. The variable itself is no object of investigation. Physicists often are interested in the variable itself. Let us show this by an example.

Kinetic energy E_{kin} is described by the formula $E_{kin} = \frac{m \cdot v^2}{2}$.

Momentum is mass times velocity: $p = m \cdot v$ so we can write alternatively:

$$E_{kin} = \frac{p^2}{2m}$$

We partially derive E_{kin} to the variable p:

$$\frac{\partial E_{kin}}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m}\right) = \frac{2p}{2m} = \frac{p}{m}$$

If we compare with the definition of momentum we get:

$$\frac{\partial E_{kin}}{\partial p} = v$$

If we partially derive $E_{kin} = \frac{m \cdot v^2}{2}$ to the variable v:

$$\frac{\partial E_{kin}}{\partial v} = \frac{\partial}{\partial v}\left(\frac{mv^2}{2}\right) = \frac{2mv}{2} = mv = p$$

The mutual dependencies $\frac{\partial E_{kin}}{\partial p} = v$ and $\frac{\partial E_{kin}}{\partial v} = p$ are what physicists are interested in.

Statistics

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called probability density function:

1. $f(x) \geq 0$
2. f is continuous
3. $\int_{-\infty}^{+\infty} f(x)dx = 1$

The probability for $a < x \leq b$:

$$P(a < x \leq b) = F(b) - F(a) = \int_a^b f(x)dx$$

Expectation value:

$$\mu = E[x] \coloneqq \begin{cases} \sum_j x_j p_j & if\ x\ is\ discrete \\ \int_{-\infty}^{+\infty} xf(x)dx & if\ x\ is\ continuous \end{cases}$$

Shifting the elements of a distribution by a constant shifts the expectation value:

$$x_j \rightarrow (x_j - a): E[X] \rightarrow E[X - a] = E[X] - a$$

Variance or expectation value of the squared deviation:

$$\sigma^2 := \begin{cases} \sum_j (x_j - \mu)^2 p_j & \mathit{if\ x\ is\ discrete} \\ \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx & \mathit{if\ x\ is\ continuous} \end{cases}$$

Standard deviation σ is the square root of the variance σ^2.

If the expectation value $E[X]$ of a distribution is 0 then the variance σ^2 is the expectation value of x^2:

$$\sigma^2 = \begin{cases} \sum_j (x_j)^2 p_j = E[x^2] & \mathit{if\ x\ is\ discrete} \\ \int_{-\infty}^{+\infty} (x)^2 f(x)dx = E[x^2] & \mathit{if\ x\ is\ continuous} \end{cases}$$