This paper deals with the time evolution of a two-level quantum system, a qubit in a real existing system.

We do this with the picture of a precessing magnetic moment (a spin up/down) and access this by both working with a static laboratory system and the rotating frame method. Both are giving the same result.

You may find more information at:

https://en.wikipedia.org/wiki/Rabi_frequency

https://quantum-abc.de/spin_flip_1.pdf

Hope I can help you with learning quantum mechanics.

This is an experimental text and may contain errors.

General two-level system

The time-dependent Hamiltonian of a precessing magnetic moment, a precessing spin:

$$H = -\vec{\mu} \cdot \vec{B} = \frac{1}{2}\hbar \begin{pmatrix} \omega_0 & \omega_R e^{-i\omega t} \\ \omega_R e^{i\omega t} & -\omega_0 \end{pmatrix}$$

This Hamiltonian describes the energy of a two-level magnetic moment in a combination of static and rotating magnetic fields:

$$\vec{B}(t) = -|\vec{B}_1|\vec{e}_x cos(\omega t) - |\vec{B}_1|\vec{e}_y sin(\omega t) + |\vec{B}_0|\vec{e}_z$$

The rotating magnetic field in the $x - y$-plane:

$$-|\vec{B}_1|\vec{e}_x cos(\omega t) - |\vec{B}_1|\vec{e}_y sin(\omega t)$$

The static field in the z-direction:

$$|\vec{B}_0|\vec{e}_z$$

Note:

$\omega_0 = \gamma|\vec{B}_0|$	Larmor frequency
ω	frequency of the driving field
$\omega_R = \gamma|\vec{B}_1|$	Rabi frequency
γ	gyromagnetic ratio
$\delta = \omega - \omega_0$	difference between frequency of the driving field and Larmor frequency
$\Omega_R = \sqrt{\delta^2 + \omega^2{}_R}$	generalized Rabi frequency.

We use the standard basis for qubits:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\downarrow\rangle = |0\rangle \coloneqq |e\rangle \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\uparrow\rangle = |1\rangle \coloneqq |g\rangle$$

Note: In spin terminology $|g\rangle$ is the ground state, $|e\rangle$ is the (first) excited state.

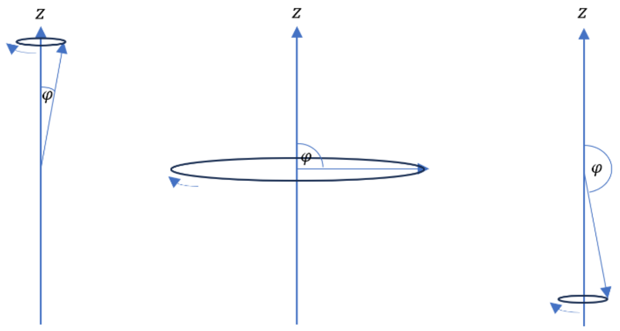
Note: In spin terminology $|\uparrow\rangle$ and $|\downarrow\rangle$ are states with $\vec{\mu}$ being oriented along the $\mp\vec{e}_z$ axis.

Note: ground state and excited state depend on the charge of the particle. For electrons the spin in direction of the magnetic field is the excited state, the spin antiparallel to the magnetic field is the ground state. For positive charges like a proton this reverses.

The general time-dependent two-level system described in probability amplitudes:

$$|\psi(t)\rangle = a_g(t)e^{i\frac{\omega_0}{2}t}|g\rangle + a_e(t)e^{-i\frac{\omega_0}{2}t}|e\rangle$$

We transform the picture of a precessing magnetic moment resp. a precessing spin around a magnetic field along the z-axis into its projection onto the z-axis.



In case the spin is precessing tightly around the z-direction the probability amplitude for spin up is nearly one, the probability amplitude for spin down is nearly zero.	In case the spin is precessing perpendicular to the z-direction the probability amplitude for spin up is the same as the probability amplitude for spin down, each equal $\sqrt{1/2}$.	In case the spin is precessing tightly around the $-z$-direction the probability amplitude for spin up is nearly zero, the probability amplitude for spin down is nearly one.

Note: Don't confuse the angle φ and the basis vectors $\binom{1}{0}$ and $\binom{0}{1}$. For $\varphi = 0$ we have 100% spin up = $\binom{1}{0}$, for $\varphi = \pi$ we have 100% spin down = $\binom{0}{1}$.

Note: Instead of using "probability amplitude" we can call this "population of a state".

We name a_g and a_e the probability amplitude of the population of ground level and excited level, both are time dependent.

According to quantum computing rules we have:

$$\left(a_g(t)e^{i\frac{\omega_0}{2}t}\right)^2 + \left(a_e(t)e^{-i\frac{\omega_0}{2}t}\right)^2 = 1$$

More precisely:

$$a_g(t)e^{i\frac{\omega_0}{2}t}a_g(t)e^{-i\frac{\omega_0}{2}t} + a_e(t)e^{-i\frac{\omega_0}{2}t}a_e(t)e^{i\frac{\omega_0}{2}t} = 1$$

This gives:

$$\left(a_g(t)\right)^2 e^{i\frac{\omega_0}{2}t}e^{-i\frac{\omega_0}{2}t} + \left(a_e(t)\right)^2 e^{-i\frac{\omega_0}{2}t}e^{i\frac{\omega_0}{2}t} = 1$$

$$\left(a_g(t)\right)^2 + \left(a_e(t)\right)^2 = 1$$

This fits with $a_g(t) = sin(t)$ and $a_e(t) = cos(t)$.

To obtain the change of probability densities $\dot{a}_g$ and $\dot{a}_e$ we solve the Schrödinger equation:

$$H|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle$$

We write $|\psi(t)\rangle$ in vector notation:

$$|\psi(t)\rangle = a_g(t)e^{i\frac{\omega_0}{2}t}\begin{pmatrix}0\\1\end{pmatrix} + a_e(t)e^{-i\frac{\omega_0}{2}t}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

We apply the Hamiltonian:	We differentiate $|\psi(t)\rangle$:
$H|\psi(t)\rangle =$ $\frac{1}{2}\hbar\begin{pmatrix}\omega_0 & \omega_R e^{-i\omega t}\\ \omega_R e^{i\omega t} & -\omega_0\end{pmatrix}\begin{pmatrix}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} =$ $\frac{1}{2}\hbar\begin{pmatrix}\omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R e^{-i\omega t}a_g(t)e^{i\frac{\omega_0}{2}t}\\ \omega_R e^{i\omega t}a_e(t)e^{-i\frac{\omega_0}{2}t} - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} =$ $\frac{1}{2}\hbar\begin{pmatrix}\omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\left(\frac{\omega_0}{2}-\omega\right)t}\\ \omega_R e^{i\left(\omega-\frac{\omega_0}{2}\right)t}a_e(t) - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$	$i\hbar\frac{d}{dt}|\psi(t)\rangle =$ $i\hbar\frac{d}{dt}\begin{pmatrix}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} =$ $i\hbar\begin{pmatrix}\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} - i\frac{\omega_0}{2}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ \dot{a}_g(t)e^{i\frac{\omega_0}{2}t} + i\frac{\omega_0}{2}a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$

We get the equation:

$$i\hbar\begin{pmatrix}\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} - i\frac{\omega_0}{2}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ \dot{a}_g(t)e^{i\frac{\omega_0}{2}t} + i\frac{\omega_0}{2}a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}\omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\left(\frac{\omega_0}{2}-\omega\right)t}\\ \omega_R e^{i\left(\omega-\frac{\omega_0}{2}\right)t}a_e(t) - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

We remove $\hbar$:

$$\begin{pmatrix}2i\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t}\\ 2i\dot{a}_g(t)e^{i\frac{\omega_0}{2}t} - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} = \begin{pmatrix}\omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\left(\frac{\omega_0}{2}-\omega\right)t}\\ \omega_R e^{i\left(\omega-\frac{\omega_0}{2}\right)t}a_e(t) - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

We treat the two equations separately:

Equation one:	Equation two:
$2i\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t} = \omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\left(\frac{\omega_0}{2}-\omega\right)t}$	$2i\dot{a}_g(t)e^{i\frac{\omega_0}{2}t} - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t} = \omega_R e^{i\left(\omega-\frac{\omega_0}{2}\right)t}a_e(t) - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}$
$2i\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} = \omega_R a_g(t)e^{i\left(\frac{\omega_0}{2}-\omega\right)t}$	$2i\dot{a}_g(t)e^{i\frac{\omega_0}{2}t} = \omega_R e^{i\left(\omega-\frac{\omega_0}{2}\right)t}a_e(t)$
$2i\dot{a}_e(t) = \omega_R a_g(t)e^{i(\omega_0-\omega)t}$	$2i\dot{a}_g(t) = \omega_R e^{i(\omega-\omega_0)t}a_e(t)$
$i\dot{a}_e(t) = \frac{1}{2}\omega_R a_g(t)e^{i(\omega_0-\omega)t}$	$i\dot{a}_g(t) = \frac{1}{2}\omega_R e^{i(\omega-\omega_0)t}a_e(t)$

What we have at this stage is the change of population or the change of probability amplitudes. This is a system of differential equations we will handle later.

Parallel, we work with the time-independent Hamiltonian H' in the rotating frame.

We compare both Hamiltonians:

The laboratory system with the time-dependent Hamiltonian H:	The rotating system with the time-independent Hamiltonian H':
$H = -\vec{\mu}\cdot\vec{B} = \frac{1}{2}\hbar\begin{pmatrix}\omega_0 & \omega_R e^{-i\omega t}\\ \omega_R e^{i\omega t} & -\omega_0\end{pmatrix}$	$H' = \frac{1}{2}\hbar\begin{pmatrix}-\delta & \omega_R\\ \omega_R & \delta\end{pmatrix}$

Note: In the rotating frame we have $\omega = 0, \delta \coloneqq \omega - \omega_0 = -\omega_0$.

To obtain the equations of motion, the change of probability densities $\dot{a}_g$ and $\dot{a}_e$ we solve the Schrödinger equation:

$$H'|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle$$

We rewrite $|\psi(t)\rangle$ in vector notation:

$$|\psi(t)\rangle = a_g(t)e^{i\frac{\omega_0}{2}t}\begin{pmatrix}0\\1\end{pmatrix} + a_e(t)e^{-i\frac{\omega_0}{2}t}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

We apply the Hamiltonian:	We differentiate $\lvert\psi(t)\rangle$:
$H'\lvert\psi(t)\rangle =$ $\frac{1}{2}\hbar\begin{pmatrix}-\delta & \omega_R\\ \omega_R & \delta\end{pmatrix}\begin{pmatrix}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} =$ $\frac{1}{2}\hbar\begin{pmatrix}-\delta a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\frac{\omega_0}{2}t}\\ \omega_R a_e(t)e^{-i\frac{\omega_0}{2}t} + \delta a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$	$i\hbar\frac{d}{dt}\lvert\psi(t)\rangle =$ $i\hbar\frac{d}{dt}\begin{pmatrix}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} =$ $i\hbar\begin{pmatrix}\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} - i\frac{\omega_0}{2}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ \dot{a}_g(t)e^{i\frac{\omega_0}{2}t} + i\frac{\omega_0}{2}a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$

We get the equation:

$$\frac{1}{2}\hbar\begin{pmatrix}-\delta a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\frac{\omega_0}{2}t}\\ \omega_R a_e(t)e^{-i\frac{\omega_0}{2}t} + \delta a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} = i\hbar\begin{pmatrix}\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} - i\frac{\omega_0}{2}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ \dot{a}_g(t)e^{i\frac{\omega_0}{2}t} + i\frac{\omega_0}{2}a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

We multiply by $\frac{2}{\hbar}$:

$$\begin{pmatrix}-\delta a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\frac{\omega_0}{2}t}\\ \omega_R a_e(t)e^{-i\frac{\omega_0}{2}t} + \delta a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} = 2i\begin{pmatrix}\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} - i\frac{\omega_0}{2}a_e(t)e^{-i\frac{\omega_0}{2}t}\\ \dot{a}_g(t)e^{i\frac{\omega_0}{2}t} + i\frac{\omega_0}{2}a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

$$\begin{pmatrix}-\delta a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\frac{\omega_0}{2}t}\\ \omega_R a_e(t)e^{-i\frac{\omega_0}{2}t} + \delta a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix} = \begin{pmatrix}2i\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t}\\ 2i\dot{a}_g(t)e^{i\frac{\omega_0}{2}t} - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}\end{pmatrix}$$

We get two equations:

$$-\delta a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\frac{\omega_0}{2}t} = 2i\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t}$$

$$\omega_R a_e(t)e^{-i\frac{\omega_0}{2}t} + \delta a_g(t)e^{i\frac{\omega_0}{2}t} = 2i\dot{a}_g(t)e^{i\frac{\omega_0}{2}t} - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}$$

Equation one:

$$-\delta a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\frac{\omega_0}{2}t} = 2i\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t}$$

$$2i\dot{a}_e(t)e^{-i\frac{\omega_0}{2}t} = -\delta a_e(t)e^{-i\frac{\omega_0}{2}t} + \omega_R a_g(t)e^{i\frac{\omega_0}{2}t} - \omega_0 a_e(t)e^{-i\frac{\omega_0}{2}t}$$

$$2i\dot{a}_e(t) = -\delta a_e(t) + \omega_R a_g(t)e^{i\omega_0 t} - \omega_0 a_e(t)$$

$$2i\dot{a}_e(t) = -a_e(t)(\delta + \omega_0) + \omega_R a_g(t)e^{i\omega_0 t}$$

$$i\dot{a}_e(t) = -\frac{1}{2}a_e(t)(\delta + \omega_0) + \frac{1}{2}\omega_R a_g(t)e^{i\omega_0 t}$$

We use the specifics of the rotating frame:

$$\omega = 0, \delta \coloneqq \omega - \omega_0 = -\omega_0$$

We get:

$$i\dot{a}_e(t) = -\frac{1}{2}a_e(t)(\delta + \omega_0) + \frac{1}{2}\omega_R a_g(t)e^{i\omega_0 t} = \frac{1}{2}\omega_R a_g(t)e^{i\omega_0 t}$$

Equation two:

$$\omega_R a_e(t)e^{-i\frac{\omega_0}{2}t} + \delta a_g(t)e^{i\frac{\omega_0}{2}t} = 2i\dot{a}_g(t)e^{i\frac{\omega_0}{2}t} - \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}$$

$$2i\dot{a}_g(t)e^{i\frac{\omega_0}{2}t} = \omega_R a_e(t)e^{-i\frac{\omega_0}{2}t} + \delta a_g(t)e^{i\frac{\omega_0}{2}t} + \omega_0 a_g(t)e^{i\frac{\omega_0}{2}t}$$

$$2i\dot{a}_g(t) = \omega_R a_e(t)e^{-i\omega_0 t} + \delta a_g(t) + \omega_0 a_g(t)$$

$$2i\dot{a}_g(t) = \omega_R a_e(t)e^{-i\omega_0 t} + \delta a_g(t)(\delta + \omega_0)$$

$$i\dot{a}_g(t) = \frac{1}{2}\omega_R a_e(t)e^{-i\omega_0 t} + \frac{1}{2}a_g(t)(\delta + \omega_0)$$

We use the specifics of the rotating frame:

$$\omega = 0, \delta \coloneqq \omega - \omega_0 = -\omega_0$$

We get:

$$i\dot{a}_g(t) = \frac{1}{2}\omega_R a_e(t)e^{-i\omega_0 t} + \frac{1}{2}a_g(t)(\delta + \omega_0) = \frac{1}{2}\omega_R a_e(t)e^{-i\omega_0 t}$$

We compare the results of the two ways to access the problem:

Static frame	Rotating frame
$i\dot{a}_g(t) = \frac{1}{2}\omega_R e^{i(\omega-\omega_0)t} a_e(t)$	$i\dot{a}_g(t) = \frac{1}{2}\omega_R a_e(t) e^{-i\omega_0 t}$
$i\dot{a}_e(t) = \frac{1}{2}\omega_R a_g(t) e^{i(\omega_0-\omega)t}$	$i\dot{a}_e(t) = \frac{1}{2}\omega_R a_g(t) e^{i\omega_0 t}$

Noting that in the rotating frame $\omega = 0$ we have the same result.

Equations of change in probability density resp. population

We rewrite $\delta = \omega - \omega_0$.

$\dot{a}_g = -i\frac{\omega_R}{2} e^{i\delta t} a_e$	$\dot{a}_e = -i\frac{\omega_R}{2} e^{-i\delta t} a_g$

This system of differential equations has the solutions (calculated by https://mathdf.com/dif/)

$$a_e = C_1 \cdot sin\left(\frac{1}{2}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) t\right) + C \cdot cos\left(\frac{1}{2}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) t\right)$$

$$a_g = C\frac{i}{\omega_R}\left(\delta - \sqrt{\delta^2 + {\omega_R}^2}\right) e^{i\delta t} \cdot sin\left(\frac{1}{2}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) t\right) + C_1\frac{i}{\omega_R}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) e^{i\delta t} \cdot cos\left(\frac{1}{2}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) t\right)$$

We use the boundary condition that the system at time $t = 0$ is in the ground state:

$$a_g(t = 0) = 1, a_e(t = 0) = 0$$

$$a_g(t = 0) = 1 = C_1\frac{i}{\omega_R}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) \to C_1 = \frac{\omega_R}{i\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right)}$$

$$a_e(t = 0) = 0 = C$$

We get the complete solution:

$$a_e = \frac{\omega_R}{i\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right)} \cdot sin\left(\frac{1}{2}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) t\right)$$

$$a_g = e^{i\delta t} \cdot cos\left(\frac{1}{2}\left(\sqrt{\delta^2 + {\omega_R}^2} - \delta\right) t\right)$$

For $(t = 0)$ the ground state is complete:

$$a_g(t = 0) = 1$$

For $(t = 0)$ the excited state is empty:

$$a_e(t = 0) = 0$$

What happens if the ground state becomes zero for the first cycle:

$$cos\left(\frac{1}{2}\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)t\right)=0\rightarrow\frac{1}{2}\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)t=\frac{\pi}{2}\rightarrow$$

$$\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)t=\pi\rightarrow t=\frac{\pi}{\sqrt{\delta^2+\omega_R{}^2}-\delta}$$

We insert into the excited state:

$$a_e=\frac{\omega_R}{i\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)}\cdot sin\left(\frac{1}{2}\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)t\right)\rightarrow$$

$$a_e=\frac{\omega_R}{i\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)}\cdot sin\left(\frac{1}{2}\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)\frac{\pi}{\sqrt{\delta^2+\omega_R{}^2}-\delta}\right)\rightarrow$$

$$a_e=\frac{\omega_R}{i\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)}\cdot sin\left(\frac{\pi}{2}\right)\rightarrow$$

$$a_e=\frac{\omega_R}{i\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)}$$

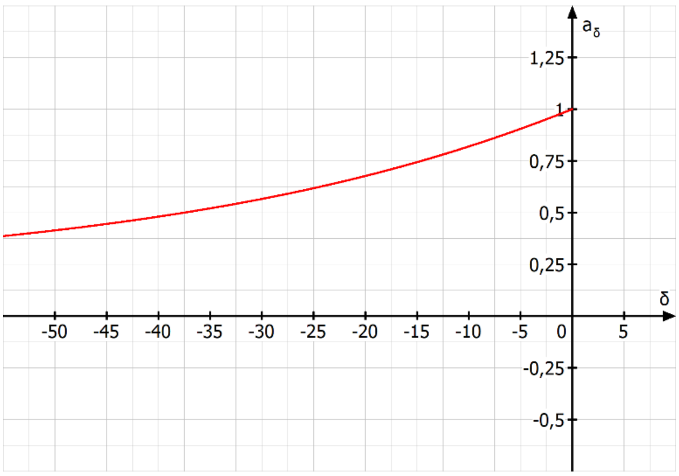
We get the absolute value of a_e:

$$|a_e|^2=\frac{\omega_R}{i\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)}\frac{\omega_R}{-i\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)}=$$

$$\frac{\omega_R{}^2}{\left(\sqrt{\delta^2+\omega_R{}^2}-\delta\right)^2}$$

Note: we need δ to be negative.

We take a look at the argument of the sin with an example $\omega_R=50$.



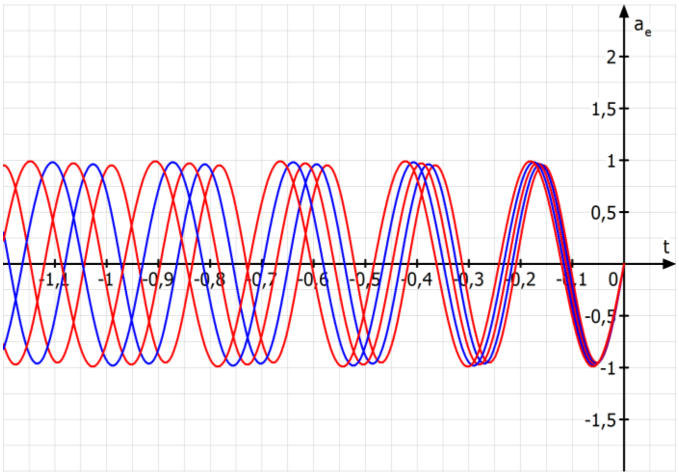
What we get is:

If the difference between the frequency of driving field and Larmor frequency is zero, we have a complete interchange between ground state and excited state.

If the difference between the frequency of driving field and Larmor frequency is not zero, the excited state will not be filled up completely.

Together with the preceding factor we get a picture for a_e:

$$a_e = \frac{\omega_R}{i\left(\sqrt{\delta^2 + \omega_R{}^2} - \delta\right)} \cdot sin\left(\frac{1}{2}\left(\sqrt{\delta^2 + \omega_R{}^2} - \delta\right)t\right)$$



Note: $\omega_R = 50, -1 < \delta < -5$