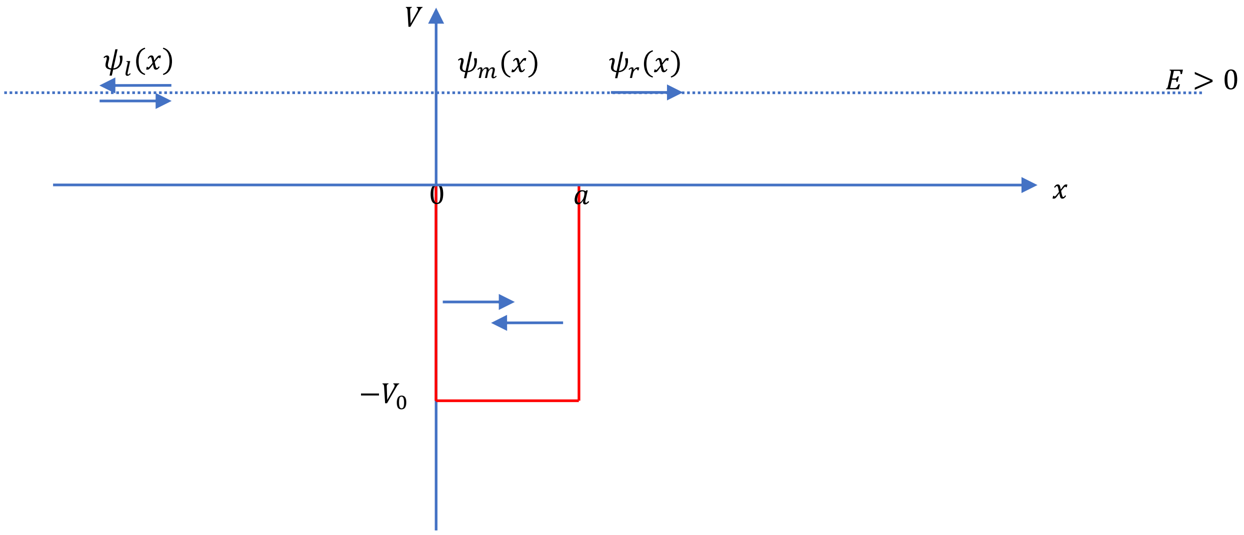
This paper deals with the Ramsauer-Townsend effect. Its main purpose is to show that calculations done with exponential functions are more straightforward compared with sin/cos functions.

Hope I can help you with learning quantum mechanics.

The Ramsauer Townsend effect

We use a finite potential well and a plane wave coming from the left.



We remember the energy. Outside the well we have energy E:

$$k^2 = \frac{2mE}{\hbar^2}$$

Inside the well we have energy $(E+V_0)$:

$$k'^2 = \frac{2m(E+V_0)}{\hbar^2}$$

$\psi_l(x)$ is a superposition of an incoming and maybe reflected wave:

$$\psi_l(x) = Ae^{ikx} + Be^{-ikx}$$

$\psi_m(x)$ is a superposition of an incoming and reflected wave above the well:

$$\psi_m(x) = Ce^{ik'x} + De^{-ik'x}$$

$\psi_r(x)$ is an outgoing wave only:

$$\psi_r(x) = Fe^{ikx}$$

Note: We use letter F to avoid mix-up with the letter E usually used for energy.

The three waves fit at the borders a and 0 together with their derivatives.

The waves:

$\psi_l(0) = Ae^{ik\cdot 0} + Be^{-ik\cdot 0} = A + B$ $\psi_m(0) = Ce^{ik'\cdot 0} + De^{-ik'\cdot 0} = C + D$	$\psi_m(a) = Ce^{ik'a} + De^{-ik'a}$ $\psi_r(a) = Fe^{ika}$

The waves fit at $x = 0$ and $x = a$:

$A + B = C + D$	$Ce^{ik'a} + De^{-ik'a} = Fe^{ika}$

The derivatives:

$$\frac{d}{dx}\psi_l(x) = \frac{d}{dx}\left(Ae^{ikx} + Be^{-ikx}\right) = ik\left(Ae^{ikx} - Be^{-ikx}\right)$$

$$\frac{d}{dx}\psi_m(x) = \frac{d}{dx}\left(Ce^{ik\prime x} + De^{-ik\prime x}\right) = ik'\left(Ce^{ik\prime x} - De^{-ik\prime x}\right)$$

$$\frac{d}{dx}\psi_r(x) = ikFe^{ikx}$$

The derivatives fit at $x = 0$ and $x = a$:

$\frac{d}{dx}\psi_l(0) = \frac{d}{dx}\psi_m(0)$	$\frac{d}{dx}\psi_m(a) = \frac{d}{dx}\psi_r(a)$
$ik\left(Ae^{ik0} - Be^{-ik0}\right) = ik'\left(Ce^{ik\prime 0} - De^{-ik\prime 0}\right)$ $ik(A - B) = ik'(C - D)$	$ik'\left(Ce^{ik'a} - De^{-ik'a}\right) = ikFe^{ika}$

We get the system of equations:

I	$A + B = C + D$
II	$Ce^{ik'a} + De^{-ik'a} = Fe^{ika}$
III	$A - B = \frac{k'}{k}(C - D)$
IV	$Ce^{ik'a} - De^{-ik'a} = \frac{k}{k'}Fe^{ika}$

We add II + IV:

$$C = \frac{F}{2}e^{ia(k-k')}\left(1 + \frac{k}{k'}\right)$$

We subtract II - IV

$$D = \frac{F}{2}e^{ia(k+k')}\left(1 - \frac{k}{k'}\right)$$

We insert the results in I and III:

$$A + B = \frac{F}{2}e^{ia(k-k')}\left(1 + \frac{k}{k'}\right) + \frac{F}{2}e^{ia(k+k')}\left(1 - \frac{k}{k'}\right)$$

$$A - B = \frac{k'}{k}\left(\frac{F}{2}e^{ia(k-k')}\left(1 + \frac{k}{k'}\right) - \frac{F}{2}e^{ia(k+k')}\left(1 - \frac{k}{k'}\right)\right)$$

We add both equations:

$$2A = \frac{F}{2}e^{ia(k-k')}\left(1 + \frac{k}{k'}\right) + \frac{F}{2}e^{ia(k+k')}\left(1 - \frac{k}{k'}\right) + \frac{k'}{k}\left(\frac{F}{2}e^{ia(k-k')}\left(1 + \frac{k}{k'}\right) - \frac{F}{2}e^{ia(k+k')}\left(1 - \frac{k}{k'}\right)\right)$$

$$\frac{4A}{F} = e^{ia(k-k')}\left(1+\frac{k}{k'}\right) + e^{ia(k+k')}\left(1-\frac{k}{k'}\right) + \frac{k'}{k}\left(e^{ia(k-k')}\left(1+\frac{k}{k'}\right) - e^{ia(k+k')}\left(1-\frac{k}{k'}\right)\right) =$$

$$e^{ia(k-k')}\left(1+\frac{k}{k'}\right) + e^{ia(k+k')}\left(1-\frac{k}{k'}\right) + \frac{k'}{k}\left(e^{ia(k-k')}\left(1+\frac{k}{k'}\right) - e^{ia(k+k')}\left(1-\frac{k}{k'}\right)\right) =$$

$$e^{ia(k-k')}\left(1+\frac{k}{k'}\right) + e^{ia(k+k')}\left(1-\frac{k}{k'}\right) + e^{ia(k-k')}\frac{k'}{k}\left(1+\frac{k}{k'}\right) - e^{ia(k+k')}\frac{k'}{k}\left(1-\frac{k}{k'}\right) =$$

$$e^{ia(k-k')}\left(\left(1+\frac{k}{k'}\right) + \frac{k'}{k}\left(1+\frac{k}{k'}\right)\right) + e^{ia(k+k')}\left(\left(1-\frac{k}{k'}\right) - \frac{k'}{k}\left(1-\frac{k}{k'}\right)\right) =$$

$$e^{ia(k-k')}\left(1+\frac{k}{k'}+\frac{k'}{k}+1\right) + e^{ia(k+k')}\left(1-\frac{k}{k'}-\frac{k'}{k}+1\right) =$$

$$e^{ia(k-k')}\left(2+\frac{k^2+k'^2}{kk'}\right) + e^{ia(k+k')}\left(2-\frac{k^2+k'^2}{kk'}\right) =$$

$$e^{ia(k-k')}\left(\frac{k^2+2kk'+k'^2}{kk'}\right) - e^{ia(k+k')}\left(\frac{k^2-2kk'+k'^2}{kk'}\right)$$

Result:

$$\frac{4A}{F} = e^{ia(k-k')}\left(\frac{(k+k')^2}{kk'}\right) - e^{ia(k+k')}\left(\frac{(k-k')^2}{kk'}\right)$$

We subtract both equations:

$$\frac{4B}{F} = e^{ia(k-k')}\left(1+\frac{k}{k'}\right) + e^{ia(k+k')}\left(1-\frac{k}{k'}\right) - \frac{k'}{k}\left(e^{ia(k-k')}\left(1+\frac{k}{k'}\right) - e^{ia(k+k')}\left(1-\frac{k}{k'}\right)\right) =$$

$$e^{ia(k-k')}\left(1+\frac{k}{k'}\right) + e^{ia(k+k')}\left(1-\frac{k}{k'}\right) - e^{ia(k-k')}\frac{k'}{k}\left(1+\frac{k}{k'}\right) + e^{ia(k+k')}\frac{k'}{k}\left(1-\frac{k}{k'}\right) =$$

$$e^{ia(k-k')}\left(\left(1+\frac{k}{k'}\right) - \frac{k'}{k}\left(1+\frac{k}{k'}\right)\right) + e^{ia(k+k')}\left(\left(1-\frac{k}{k'}\right) + \frac{k'}{k}\left(1-\frac{k}{k'}\right)\right) =$$

$$e^{ia(k-k')}\left(1+\frac{k}{k'}-\frac{k'}{k}-1\right) + e^{ia(k+k')}\left(1-\frac{k}{k'}+\frac{k'}{k}-1\right) =$$

$$e^{ia(k-k')}\left(\frac{k^2-k'^2}{kk'}\right) + e^{ia(k+k')}\left(\frac{k'^2-k^2}{kk'}\right) =$$

$$e^{ia(k-k')}\left(\frac{k^2-k'^2}{kk'}\right) - e^{ia(k+k')}\left(\frac{k^2-k'^2}{kk'}\right)$$

Result:

$$\frac{4B}{F} = \left(e^{ia(k-k')} - e^{ia(k+k')}\right)\left(\frac{k^2-k'^2}{kk'}\right)$$

We calculate the ratio of the amplitudes of incoming wave A to outgoing wave F, ignoring for simplicity the factor 4:

$$\frac{A}{F} = e^{ia(k-k')}\left(2 + \frac{k^2 + k'^2}{kk'}\right) + e^{ia(k+k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right)$$

$$\left|\frac{A}{F}\right|^2 = \left(e^{ia(k-k')}\left(2 + \frac{k^2 + k'^2}{kk'}\right) + e^{ia(k+k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right)\right)\left(e^{-ia(k-k')}\left(2 + \frac{k^2 + k'^2}{kk'}\right) + e^{-ia(k+k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right)\right) =$$

$$\left(e^{ia(k-k')}\left(2 + \frac{k^2 + k'^2}{kk'}\right) + e^{ia(k+k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right)\right)\left(e^{ia(k'-k)}\left(2 + \frac{k^2 + k'^2}{kk'}\right) + e^{ia(-k-k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right)\right) =$$

$$e^{ia(k-k')}\left(2 + \frac{k^2 + k'^2}{kk'}\right)e^{ia(k'-k)}\left(2 + \frac{k^2 + k'^2}{kk'}\right) + e^{ia(k-k')}\left(2 + \frac{k^2 + k'^2}{kk'}\right)e^{ia(-k-k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right) + e^{ia(k+k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right)e^{ia(k'-k)}\left(2 + \frac{k^2 + k'^2}{kk'}\right) + e^{ia(k+k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right)e^{ia(-k-k')}\left(2 - \frac{k^2 + k'^2}{kk'}\right) =$$

$$\left(2 + \frac{k^2 + k'^2}{kk'}\right)^2 + e^{-ia2k'}\left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right) + e^{ia2k'}\left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right) + \left(2 - \frac{k^2 + k'^2}{kk'}\right)^2 =$$

$$\left(2 + \frac{k^2 + k'^2}{kk'}\right)^2 + \left(e^{-ia2k'} + e^{ia2k'}\right)\left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right) + \left(2 - \frac{k^2 + k'^2}{kk'}\right)^2 =$$

$$2\left(4 + \left(\frac{k^2 + k'^2}{kk'}\right)^2\right) + \left(e^{-ia2k'} + e^{ia2k'}\right)\left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right) =$$

$$2\left(4 + \left(\frac{k^2 + k'^2}{kk'}\right)^2\right) + \left(Re\left(e^{ia2k'}\right)\right) \cdot 2 \cdot \left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right)$$

Result (with the correct factor added):

$$\left|\frac{A}{F}\right|^2 = \frac{4 + \left(\frac{k^2 + k'^2}{kk'}\right)^2 + \left(Re\left(e^{ia2k'}\right)\right) \cdot \left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right)}{8}$$

We check:

$k = k'$ meaning there is no Potential:

$$\left|\frac{A}{F}\right|^2 = \frac{4 + \left(\frac{k^2 + k^2}{kk}\right)^2 + \left(Re\left(e^{ia2k}\right)\right) \cdot \left(4 - \left(\frac{k^2 + k^2}{kk}\right)^2\right)}{8} = \frac{8}{8} = 1$$

Result: We get a transmitted wave F only, no reflection.

We recheck this by calculating $\frac{B}{F}$ for this case:

$$4\frac{B}{F} = 4\left(\left(e^{ia(k-k)} - e^{ia(k+k)}\right)\left(\frac{k^2 - k^2}{kk}\right)\right) = 0$$

We want to plot the transmission ratio as a function of V_0.

$\left|\frac{A}{F}\right|^2$ could be problematic if F becomes zero, so we plot the reverse:

$$\left|\frac{F}{A}\right|^2 = \frac{8}{4 + \left(\frac{k^2 + k'^2}{kk'}\right)^2 + \left(Re\left(e^{ia2k'}\right)\right) \cdot \left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right)}$$

We use:

$$k^2 = \frac{2mE}{\hbar^2} := 1$$

$$k = 1$$

$$k'^2 = \frac{2m(E + V_0)}{\hbar^2} = \frac{2mE}{\hbar^2} + \frac{2mV_0}{\hbar^2} = k^2 + \Delta k = 1 + \Delta k$$

$$k' = \sqrt{1 + \Delta k}$$

$$a = 1$$

We get:

$$\left|\frac{F}{A}\right|^2 = \frac{8}{4 + \left(\frac{k^2 + k'^2}{kk'}\right)^2 + \left(Re\left(e^{ia2k'}\right)\right) \cdot \left(4 - \left(\frac{k^2 + k'^2}{kk'}\right)^2\right)} =$$

$$\frac{8}{4 + \left(\frac{1 + 1 + \Delta k}{\sqrt{1 + \Delta k}}\right)^2 + \left(Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right) \cdot \left(4 - \left(\frac{1 + 1 + \Delta k}{\sqrt{1 + \Delta k}}\right)^2\right)} =$$

$$\frac{8}{4 + \left(\frac{2 + \Delta k}{\sqrt{1 + \Delta k}}\right)^2 + \left(Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right) \cdot \left(4 - \left(\frac{2 + \Delta k}{\sqrt{1 + \Delta k}}\right)^2\right)} =$$

$$\frac{8}{4 + \frac{(2 + \Delta k)^2}{\sqrt{1 + \Delta k}} + \left(Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right) \cdot \left(4 - \frac{(2 + \Delta k)^2}{\sqrt{1 + \Delta k}}\right)} =$$

$$\frac{8}{\frac{4\sqrt{1+\Delta k}+(2+\Delta k)^2}{\sqrt{1+\Delta k}}+\left(Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right)\cdot\left(\frac{4\sqrt{1+\Delta k}-(2+\Delta k)^2}{\sqrt{1+\Delta k}}\right)}=$$

$$\frac{8\cdot\sqrt{1+\Delta k}}{4\sqrt{1+\Delta k}+(2+\Delta k)^2+\left(Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right)4\sqrt{1+\Delta k}-\left(Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right)(2+\Delta k)^2}=$$

$$\frac{8\cdot\sqrt{1+\Delta k}}{4\sqrt{1+\Delta k}\left(1+Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right)+(2+\Delta k)^2\left(1-Re\left(e^{ia2\sqrt{1+\Delta k}}\right)\right)}=$$

$$\frac{8\cdot\sqrt{1+\Delta k}}{4\sqrt{1+\Delta k}\left(1+cos\left(2\sqrt{1+\Delta k}\right)\right)+(2+\Delta k)^2\left(1-cos\left(2\sqrt{1+\Delta k}\right)\right)}$$

The plot: