This short paper deals with the flux of wave functions (free particles) while scattering. The main purpose is working with the wave functions, not going into detail of the scattering itself. For this you might see the following links:

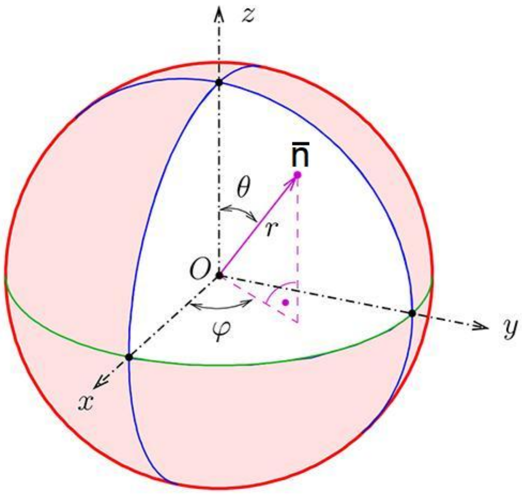
https://www.tcm.phy.cam.ac.uk/~bds10/aqp/lec20-21_compressed.pdf

https://ocw.mit.edu/courses/8-06-quantum-physics-iii-spring-2018/45dbd7038c3e969491eceeae86c44d42_MIT8_06S18ch7.pdf

Hope I can help you learning quantum mechanics.

The simplest scattering experiment: a plane wave moving on the z-axis impinging on localized potential at the origin, V (r).

The space is oriented in the usual way:



We imagine that we use a constant beam of particles, so the time dependency of the wave function results in a global phase $e^{-i\frac{E}{\hbar}t}$ we can omit in our calculations.

Both the incoming and the scattered wave must be present, so we get the following wave-function in position representation:

$$\psi(\vec{r}) = e^{ikz} + \frac{f(\theta)}{r} e^{ikr} = \psi_1(\vec{r}) + \psi_2(\vec{r})$$

$\psi_1(\vec{r}) = e^{ikz}$ represents the incoming wave function along the z-axis.

$\psi_2(\vec{r}) = \frac{f(\theta)}{r} e^{ikr}$ represents the outgoing wave function.

We assume $r \gg 1$.

The amplitude $f(\theta)$ for the scattered particles is θ-dependent but not φ-dependent. We will not investigate $f(\theta)$, a complex function of θ.

The probability current $\hat{J}$ for wave functions:

$$\hat{J} = \frac{\hbar}{m} \cdot Im(\psi^* \nabla \psi)$$

Note: Im denotes the imaginary part of a complex function/value, $*$ denotes complex conjugation.

We get:

$$\hat{J} = \frac{\hbar}{m} \cdot Im\left(\left(e^{-ikz} + \frac{f(\theta)^*}{r} e^{-ikr}\right) \nabla \left(e^{ikz} + \frac{f(\theta)}{r} e^{ikr}\right)\right) =$$

$$\frac{\hbar}{m} \cdot Im\left(e^{-ikz} \nabla e^{ikz} + e^{-ikz} \nabla \frac{f(\theta)}{r} e^{ikr} + \frac{f(\theta)^*}{r} e^{-ikr} \nabla e^{ikz} + \frac{f(\theta)^*}{r} e^{-ikr} \nabla \frac{f(\theta)}{r} e^{ikr}\right)$$

We handle this in three parts:

z-part $\hat{J}_z$	Interference part $\hat{J}_{int}$	spherical part $\hat{J}_s$
$\hat{J}_z = \frac{\hbar}{m} \cdot Im(e^{-ikz}\nabla e^{ikz})$	$\hat{J}_{int} = \frac{\hbar}{m} \cdot Im\left(e^{-ikz}\nabla\frac{f(\theta)}{r}e^{ikr} + \frac{f(\theta)^*}{r}e^{-ikr}\nabla e^{ikz}\right)$	$\hat{J}_s = \frac{\hbar}{m} \cdot Im\left(\frac{f(\theta)^*}{r}e^{-ikr}\nabla\frac{f(\theta)}{r}e^{ikr}\right)$

$\hat{J}_z$ represents the contribution from the incoming plane wave, $\hat{J}_3$ the contribution from the deflected spherical waves, $\hat{J}_2$ is the interference between incoming and outgoing waves.

z-part $\hat{J}_z$

$$\hat{J}_z = \frac{\hbar}{m} \cdot Im(e^{-ikz}\nabla e^{ikz})$$

$$e^{-ikz}\nabla e^{ikz} = e^{-ikz}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}e^{ikz} = e^{-ikz}\begin{pmatrix}0\\0\\ike^{ikz}\end{pmatrix} = \begin{pmatrix}0\\0\\ik\end{pmatrix} = ik\vec{z}$$

We get:

$$\hat{J}_z = \frac{\hbar}{m} \cdot Im(ik\vec{z}) = \frac{\hbar k}{m}\vec{z}$$

The flux Φ_z over a sphere of radius R:

$$\Phi_z = \oiint_R \hat{J}_1 \cdot d\vec{a} =$$

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{\hbar k}{m}\vec{z}\cdot\vec{r}R^2 sin(\theta)d\theta d\varphi =$$

$$\frac{\hbar k R^2}{m}\int_{\varphi=0}^{2\pi}d\varphi\int_{\theta=0}^{\pi}sin(\theta)cos(\theta)d\theta =$$

$$2\pi\frac{\hbar k R^2}{m}\left[\frac{1}{2}sin^2(\theta)\right]_0^{\pi} = 0$$

Note: $\vec{z}\cdot\vec{r} = cos(\theta)$

Interpretation: If we send a stream into a sphere and it goes out of it the content of the sphere will remain unchanged, so the flux will be zero if the incoming rate equals the outgoing rate.

Spherical part $\hat{J}_s$

$$\hat{J}_s = \frac{\hbar}{m} \cdot Im\left(\frac{f(\theta)^*}{r}e^{-ikr}\cdot\nabla\frac{f(\theta)}{r}e^{ikr}\right)$$

$$\vec{r}\cdot\hat{J}_s = \frac{\hbar}{m} \cdot Im\left(\frac{f(\theta)^*}{r}e^{-ikr}\cdot\vec{r}\cdot\nabla\frac{f(\theta)}{r}e^{ikr}\right) =$$

$$\frac{\hbar}{m}\cdot\frac{|f(\theta)|^2}{r}\cdot Im\left(e^{-ikr}\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right) =$$

Note: $\vec{r}\cdot\nabla = \frac{\partial}{\partial r}$

$$\frac{\hbar}{m} \cdot \frac{|f(\theta)|^2}{r} \cdot Im\left(e^{-ikr} \frac{ikre^{ikr} - e^{ikr}}{r^2}\right) =$$

$$\frac{\hbar}{m} \cdot \frac{|f(\theta)|^2}{r} \cdot Im\left(\frac{ikr - 1}{r^2}\right) =$$

$$\frac{\hbar}{m} \cdot \frac{|f(\theta)|^2}{r} \cdot Im\left(\frac{ik}{r}\right) =$$

Note: $Im\left(\frac{1}{r^2}\right) = 0$

$$\frac{\hbar k}{m} \cdot \frac{|f(\theta)|^2}{r^2}$$

The flux Φ_s over a sphere of radius r:

$$\Phi_s = \oiint_R \hat{J}_s \cdot d\vec{a} = \oiint_R (\hat{J}_s \cdot \vec{r}) r^2 d\Omega = \frac{\hbar k}{m} \cdot \oiint_R \frac{|f(\theta)|^2}{r^2} r^2 d\Omega =$$

$$\frac{\hbar k}{m} \cdot \oiint_R |f(\theta)|^2 d\Omega$$

We rewrite the angular element $d\Omega$:

$$d\Omega = sin(\theta) d\theta d\varphi$$

We get:

$$\Phi_s = \frac{\hbar k}{m} \cdot \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} |f(\theta)|^2 sin(\theta) d\theta d\varphi =$$

$$\frac{2\pi\hbar k}{m} \cdot \int_{\theta=0}^{\pi} |f(\theta)|^2 sin(\theta) d\theta$$

Interference part

The interference term $\hat{J}_{int}$ is given by:

$$\hat{J}_{int} = \frac{\hbar}{m} \cdot Im\left(e^{-ikz} \nabla \frac{f(\theta)}{r} e^{ikr} + \frac{f(\theta)^*}{r} e^{-ikr} \nabla e^{ikz}\right)$$

We compute:

$$\vec{r} \cdot \hat{J}_{int} = \vec{r} \cdot \frac{\hbar}{m} \cdot Im\left(e^{-ikz} \nabla \frac{f(\theta)}{r} e^{ikr} + \frac{f(\theta)^*}{r} e^{-ikr} \nabla e^{ikz}\right) =$$

$$\frac{\hbar}{m} \cdot Im\left(e^{-ikz} \vec{r} \cdot \nabla \frac{f(\theta)}{r} e^{ikr} + \frac{f(\theta)^*}{r} e^{-ikr} \vec{r} \cdot \nabla e^{ikz}\right)$$

We calculate separately:	
$e^{-ikz}\vec{r}\cdot\nabla\left(\frac{f(\theta)}{r}e^{ikr}\right)=$ $f(\theta)e^{-ikz}\vec{r}\cdot\nabla\left(\frac{e^{ikr}}{r}\right)=$ $f(\theta)e^{-ikz}\frac{ikre^{ikr}-e^{ikr}}{r^2}=$ $f(\theta)e^{-ikz}e^{ikr}\frac{ikr-1}{r^2}=$ $\frac{f(\theta)}{r}e^{ik(r-z)}\left(ik-\frac{1}{r}\right)$	$\frac{f(\theta)^*}{r}e^{-ikr}\vec{r}\cdot\nabla e^{ikz}=$ $\frac{f(\theta)^*}{r}e^{-ikr}\vec{r}\cdot\nabla e^{ikz}=$ $\frac{f(\theta)^*}{r}e^{-ikr}\vec{r}\cdot\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}e^{ikz}=$ $\frac{f(\theta)^*}{r}e^{-ikr}\vec{r}\cdot\begin{pmatrix}0\\0\\ike^{ikz}\end{pmatrix}=$ $\frac{f(\theta)^*}{r}e^{-ikr}\vec{r}\cdot\vec{z}\;ike^{ikz}$
As we use $R\gg 1$ we concentrate on:	
$ik\frac{f(\theta)}{r}e^{ik(r-z)}$	
We use:	
$z=r\,cos(\theta)$	$\vec{r}\cdot\vec{z}=cos(\theta)$
We get:	
$e^{-ikz}\vec{r}\cdot\nabla\left(\frac{f(\theta)}{r}e^{ikr}\right)\approx ik\frac{f(\theta)}{r}e^{ikr\left(1-cos(\theta)\right)}$	$\frac{f(\theta)^*}{r}e^{-ikr}\vec{r}\cdot\nabla e^{ikz}=$ $\frac{f(\theta)^*}{r}ik\,cos(\theta)e^{-ikr\left(1-cos(\theta)\right)}$

We combine the results:

$$\vec{r}\cdot\hat{J}_{int}=\frac{\hbar}{m}\cdot Im\left(e^{-ikz}\vec{r}\cdot\nabla\frac{f(\theta)}{r}e^{ikr}+\frac{f(\theta)^*}{r}e^{-ikr}\vec{r}\cdot\nabla e^{ikz}\right)\approx$$

$$\frac{\hbar k}{mr}\cdot Im\left(i\left(f(\theta)e^{ikr\left(1-cos(\theta)\right)}+f(\theta)^*\,cos(\theta)e^{-ikr\left(1-cos(\theta)\right)}\right)\right)=$$

$$\frac{\hbar k}{mr}\cdot\left(f(\theta)e^{ikr\left(1-cos(\theta)\right)}+f(\theta)^*cos(\theta)e^{-ikr\left(1-cos(\theta)\right)}\right)$$

Remark:

The local flux $\vec{r}\cdot\hat{J}_{int}$ appears to go like $\frac{1}{r}$ but integrating the sphere over θ gives another factor $\frac{1}{r}$ keeping the total flux finite.