This paper works through the problem of the spin of an electron in a magnetic field.

We will look at:

- a stationary magnetic field in z-direction
- a field constantly pointing in z-direction but its strength changing from $+max$ to $-max$
- a field with constant z-component rotating in the $x - y$ plane

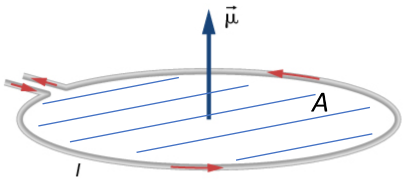
The text follows a course on MIT of B. Zwiebach:

https://ocw.mit.edu/courses/physics/8-05-quantum-physics-ii-fall-2013/lecture-notes/MIT8_05F13_Chap_07.pdf

Hope I can help you with learning quantum mechanics.

Contents

Classic magnetic moment



A current on a closed loop induces a magnetic moment.

The vector of the magnetic moment $\vec{\mu}$ is proportional to the current I in the loop and the vector area $\vec{A}$ of the loop:

$$\vec{\mu} = I \cdot \vec{A}$$

For a planar loop, the vector area is a vector normal to the loop plane with length equal to the value of the area.

The direction of the normal vector is determined by the right-hand rule.

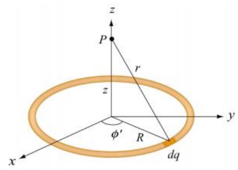
The product $\vec{\mu}B$ of the magnitude $\vec{\mu}$ of the magnetic moment times the magnetic field B has units of energy:

$$[\mu] = \frac{erg}{gauss} \ or \ \frac{Joule}{Tesla}$$

Any rotating charge distribution results in a magnetic moment.

Any rotating mass has angular momentum too.

The magnetic moment and the angular momentum are proportional to each other.



Consider a ring with radius R that has uniform charge distribution and a total charge Q. The ring itself is not conducting, the charge fixed onto the ring.

The ring is rotating about an axis perpendicular to the plane of the ring, going through its center.

The tangential velocity be v.

The current at the loop is equal to the charge density dq times the velocity:

$$I = dq \cdot v = \frac{Q}{2\pi R} \cdot v$$

Note: this formula not valid for $R = 0$.

In the situation shown we can omit the vector arrows because all directions are neatly arranged. We get the magnitude $\vec{\mu}$ of the dipole moment of the loop:

$$\mu = I \cdot A = \frac{Q}{2\pi R} \cdot v \cdot R^2\pi = \frac{1}{2} QRv$$

The non-conduction Ring has mass M and an angular momentum L:

$$L = M \cdot Rv \rightarrow Rv = \frac{L}{M}$$

We get the ratio:

$$\mu = \frac{QL}{2M}$$

Note: this formula not valid for $M = 0$.

Note: the ratio is independent of the radius of the ring and the velocity it is rotating.

Quantum mechanics, static magnetic field

The electron is viewed as an elementary particle with zero size.

We introduce the Bohr-magneton μ_B:

$$\mu_B := \frac{e\hbar}{2m_e} = 5.78 \times 10^{-5} \frac{eV}{Tesla}$$

Note: e is the charge of the electron.

We write the magnetic moment of an electron:

$$\vec{\mu}_e = -g\mu_B \frac{\hat{S}}{\hbar}$$

Note: dipole moment and angular momentum are antiparallel for electrons.

Note: $\hat{S}$ is the spin of the electron, g is the Landé g-factor with value ≈ 2 for an electron.

We simplify notation by defining the constant γ that holds g-factor, mass, and charge:

$$\gamma := \frac{ge}{2m_e}$$

With this constant we write the magnetic moment for electrons:

$$\vec{\mu} = -\gamma\hat{S}$$

Note: S is the three-component spin operator:

$$\hat{S} := \hat{S}_x + \hat{S}_y + \hat{S}_z$$

We insert the particle in the magnetic field $\vec{B}$. The Hamiltonian $\widehat{H}$ for the spin:

$$\widehat{H} = -\vec{\mu} \cdot \vec{B} = -\gamma\vec{B} \cdot \vec{S} = -\gamma\left(B_x\hat{S}_x + B_y\hat{S}_y + B_z\hat{S}_z\right)$$

The magnetic field is static along the z-axis:

$$\widehat{H} = -\gamma\left(B_z\hat{S}_z\right)$$

For a time-independent Hamiltonian, we have an associated time evolution unitary operator:

$$\widehat{U}(t,0) = e^{-\frac{i\widehat{H}}{\hbar}t} = e^{-\frac{i}{\hbar}-\gamma B_z\hat{S}_z t}$$

A prototype for a 3D rotation operator we find in Griffiths:

$$\widehat{R}_n(\varphi) = e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\vec{S}} = e^{-\frac{i}{\hbar}\varphi\hat{S}_n}$$

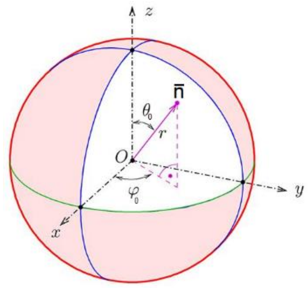
Note: φ is the rotation angle of a right-handed rotation, $\vec{n}$ is the axis of the rotation, $\hat{S}$ a set of angular momentum operators.

This resembles our unitary operator:

$$e^{-\frac{i}{\hbar}-\gamma B_z\hat{S}_z t} \sim e^{-\frac{i}{\hbar}\varphi\hat{S}_n}$$

We conclude that $\widehat{U}(t,0)$ should generate a rotation by the angle $-\gamma B_z t$ around the z-axis.

We check this explicitly.



At time $t = 0$ the spin state is pointing along the direction specified by angles θ_0, φ_0:

$$|\Psi, 0\rangle = cos\frac{\theta_0}{2}|+\rangle + sin\frac{\theta_0}{2}e^{i\varphi_0}|-\rangle$$

The magnetic field is pointing to the z-direction. We apply the Hamiltonian $\hat{H} = -\gamma(B_z\hat{S}_z)$ to the basis states $|+\rangle \coloneqq \begin{pmatrix}1\\0\end{pmatrix}$ and $|-\rangle \coloneqq \begin{pmatrix}0\\1\end{pmatrix}$:

$$\hat{H}|+\rangle = -\gamma B_z\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = -\gamma B_z\frac{\hbar}{2}|+\rangle$$

$$\hat{H}|-\rangle = -\gamma B_z\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = +\gamma B_z\frac{\hbar}{2}|-\rangle$$

We apply the time evolution unitary operator:

$$|\Psi, t\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\Psi, 0\rangle =$$

$$e^{-\frac{i\hat{H}t}{\hbar}}\left(cos\frac{\theta_0}{2}|+\rangle + sin\frac{\theta_0}{2}e^{i\varphi_0}|-\rangle\right) =$$

$$cos\frac{\theta_0}{2}e^{-\frac{i\hat{H}t}{\hbar}}|+\rangle + sin\frac{\theta_0}{2}e^{-\frac{i\hat{H}t}{\hbar}}e^{i\varphi_0}|-\rangle =$$

$$cos\frac{\theta_0}{2}e^{-\left(\frac{\frac{-i\gamma B_z\hbar t}{2}}{\hbar}\right)}|+\rangle + sin\frac{\theta_0}{2}e^{-\left(\frac{\frac{i\gamma B_z\hbar t}{2}}{\hbar}\right)}e^{i\varphi_0}|-\rangle =$$

$$cos\frac{\theta_0}{2}e^{\frac{i\gamma B_z t}{2}}|+\rangle + sin\frac{\theta_0}{2}e^{-\frac{i\gamma B_z t}{2}}e^{i\varphi_0}|-\rangle =$$

$$e^{\frac{i\gamma B_z t}{2}}\left(cos\frac{\theta_0}{2}|+\rangle + sin\frac{\theta_0}{2}e^{-i\gamma B_z t}e^{i\varphi_0}|-\rangle\right) =$$

$$e^{\frac{i\gamma B_z t}{2}}\left(cos\frac{\theta_0}{2}|+\rangle + sin\frac{\theta_0}{2}e^{i(\varphi_0-\gamma B_z t)}|-\rangle\right)$$

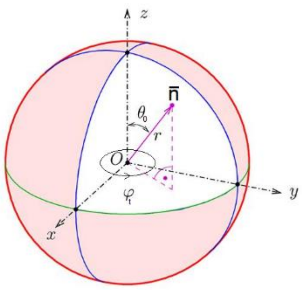
We ignore the overall phase $e^{\frac{i\gamma B_z t}{2}}$ and recognize the spin state as the state corresponding to the vector $\vec{n}_t$, defined by angles θ and φ:

$$\theta(t) = \theta_0$$

$$\varphi(t) = \varphi_0 - \gamma B_z t$$

θ is constant over time. We get a rotation around the z-axis with constant angular speed:

$$\frac{\partial}{\partial t}\varphi(t) = -\gamma B_z$$



Conclusion:
spin states in a magnetic field precess in the same way as classical magnetic dipoles.

We can ask: if the spin initially points into the x-direction – how long does it last till the spin points into the opposite $(-x)$-direction?

In this case $\varphi_0 = 0$.

$$\varphi(t) = \gamma B_z t = \pi \rightarrow t = \frac{\pi}{\gamma B_z}$$

Note: γB_z often called ω_0.

Magnetic field with constant direction but varying in strength

We have a magnetic field constantly pointing in z-direction. The strength of the field is oscillating between a maximum and a minimum:

$$\vec{B} = B_0 \cdot cos(\omega t)\overrightarrow{e_z}$$

The spin at time $t = 0$ is pointing in x-direction:

$$|\psi, t_0\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$$

The Hamiltonian $\hat{H} = -\gamma\vec{B} \cdot \hat{S}$ in this case is time dependent. The Hamiltonians of different times commute (they are independent):

$$\left[\hat{H}(t_0), \hat{H}(t_1)\right] = 0$$

The Hamiltonian:

$$\hat{H} = -\gamma\vec{B} \cdot \vec{S} = -\gamma B_0 \cdot cos(\omega t)\overrightarrow{S_z}$$

For convenience we set $\gamma B_0 \coloneqq \omega_0$ and write:

(*) $$\hat{H} = -\omega_0 \cdot cos(\omega t)\overrightarrow{S_z}$$

Approach with power series expansion

Time evolution

Time evolution in a quantum system is performed by a unitary operator $\hat{U}(t_0, t)$:

$$|\psi, t\rangle = \hat{U}(t, t_0)|\psi, t_0\rangle \quad \forall t, t_0$$

Unitary operator properties:

$$\hat{U}(t_0, t_0) = id$$

$$\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\hat{U}(t_1, t_0)$$

$$\hat{U}(t_0, t)\hat{U}(t, t_0) = id$$

$$\hat{U}(t_0, t) = \hat{U}^{-1}(t, t_0) = \hat{U}^{\dagger}(t, t_0)$$

Time evolution of a state vector:

$$|\psi, t\rangle = \hat{U}(t, t_0)|\psi, t_0\rangle \quad \forall t, t_0$$

We derive with respect to time:

$$\frac{\partial|\psi(t)\rangle}{\partial t} = \frac{\partial\hat{U}(t, t_0)}{\partial t}|\psi, t_0\rangle =$$

$$\frac{\partial \widehat{U}(t,t_0)}{\partial t}\widehat{U}(t_0,t)|\psi,t\rangle =$$

$$\frac{\partial \widehat{U}(t,t_0)}{\partial t}\widehat{U}^{\dagger}(t,t_0)|\psi,t\rangle$$

We rename:

$$\frac{\partial \widehat{U}(t,t_0)}{\partial t}\widehat{U}^{\dagger}(t,t_0) \rightarrow \widehat{\Lambda}(t,t_0)$$

Note:

$$\widehat{\Lambda}^{\dagger}(t,t_0) = \widehat{U}(t,t_0)\frac{\partial \widehat{U}^{\dagger}(t,t_0)}{\partial t}$$

Properties of $\widehat{\Lambda}(t,t_0)$:

$\widehat{\Lambda}(t,t_0)$ is antihermitian:

$$\widehat{\Lambda}(t,t_0) = -\widehat{\Lambda}^{\dagger}(t,t_0)$$

$\widehat{\Lambda}(t,t_0)$ is independent of t_0:

$$\widehat{\Lambda}(t,t_0) \coloneqq \widehat{\Lambda}(t)$$

With this we write:

$$\frac{\partial|\psi(t)\rangle}{\partial t} = \widehat{\Lambda}(t)|\psi,t\rangle$$

We multiply $\widehat{\Lambda}(t)$ by $\mathrm{i}\hbar$ and get the Hamiltonian:

$$\widehat{\mathrm{H}}(t) \coloneqq \mathrm{i}\hbar\widehat{\Lambda}(t) = \mathrm{i}\hbar\frac{\partial \widehat{U}(t,t_0)}{\partial t}\widehat{U}^{\dagger}(t,t_0)$$

The operator $\widehat{\mathrm{H}}(t)$ is Hermitian because $\widehat{\Lambda}(t)$ was anti-hermitian.

We end up with the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \widehat{\mathrm{H}}(t)|\psi(t)\rangle$$

Hamiltonian

If we know the unitary time evolution, we can derive the Hamiltonian.

We can go this way backwards and calculate the time evolution operator from the Hamiltonian $\widehat{H}$.
We use the Hamiltonian (*):

$$\widehat{H}(t) = -\omega_0 cos(\omega t)\hat{S}_z$$

We get the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\psi(t)}{\partial t} = \widehat{\mathrm{H}}(t)|\psi(t)\rangle \rightarrow$$

$$\mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} = -\omega_0 cos(\omega t)\hat{S}_z\psi(t)$$

We calculate the time evolution operator from the Hamiltonian.

We have:

$$\hat{H}(t) = \mathrm{i}\hbar \frac{\partial \hat{U}(t,t_0)}{\partial t} \hat{U}^\dagger(t,t_0) \to \hat{H}(t)\hat{U}(t,t_0) = \mathrm{i}\hbar \frac{\partial \hat{U}(t,t_0)}{\partial t}$$

This is a differential equation for $\hat{U}(t,t_0)$. Since there is no confusion with time derivatives, we write:

$$\frac{d\hat{U}(t,t_0)}{dt} = -\frac{i}{\hbar}\hat{H}(t)\hat{U}(t,t_0)$$

If we view operators as matrices, this is a differential equation for the matrix U.

The Hamiltonians for different times commute:

$$\left[\hat{H}(t_1), \hat{H}(t_2)\right] = 0$$

The time evolution operator then is given:

$$\hat{U}(t,t_0) = exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(\tau)d\tau\right)$$

We concentrate on the exponential:

$$\hat{R}(t) \coloneqq -\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(\tau)d\tau$$

$$\dot{\hat{R}}(t) = -\frac{i}{\hbar}\hat{H}(t)$$

We get:

$$\hat{U}(t) = e^{\hat{R}(t)}$$

The power series:

$$\hat{U}(t) = 1 + \hat{R}(t) + \frac{1}{2}\hat{R}(t)\hat{R}(t) + \frac{1}{3!}\hat{R}(t)\hat{R}(t)\hat{R}(t) + \cdots$$

The time derivative:

$$\dot{\hat{U}}(t) = \dot{\hat{R}}(t) + \frac{1}{2}\left(\dot{\hat{R}}(t)\hat{R}(t) + \hat{R}(t)\dot{\hat{R}}(t)\right) + \frac{1}{3!}\left(\dot{\hat{R}}(t)\hat{R}(t)\hat{R}(t) + \hat{R}(t)\dot{\hat{R}}(t)\hat{R}(t) + \hat{R}(t)\hat{R}(t)\dot{\hat{R}}(t)\right) =;$$

Note: $\hat{R}(t)\ and\ \dot{\hat{R}}(t)$ are commuting.

$$= \dot{\hat{R}}(t) + \dot{\hat{R}}(t)\hat{R}(t) + \frac{1}{2!}\dot{\hat{R}}(t)\hat{R}(t)\hat{R}(t) + \cdots \to$$

$$\frac{d\hat{U}(t)}{dt} = \dot{\hat{R}}(t)e^{\hat{R}(t)}$$

With this we can write:

$$\frac{d\hat{U}(t)}{dt} = -\frac{i}{\hbar}\hat{H}(t)exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\tau)d\tau\right)$$

We look at the exponent:

$$-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\tau)d\tau = -\frac{i}{\hbar}\int_{t_0}^{t}-\omega_0 cos(\omega\tau)\hat{S}_z d\tau =$$

$$-\frac{i}{\hbar}\left[-\frac{\omega_0}{\omega}sin(\omega\tau)\hat{S}_z\right]_{t_0}^{t} =;$$

We set $t_0 = 0$:

$$-\frac{i}{\hbar}\left(-\frac{\omega_0}{\omega}sin(\omega t)\hat{S}_z\right) = \frac{i\omega_0}{\hbar\omega}sin(\omega t)\hat{S}_z$$

The complete derivative:

$$\frac{d\hat{U}(t)}{dt} = -\frac{i}{\hbar}\hat{H}(t)exp\left(\frac{i\omega_0}{\hbar\omega}sin(\omega t)\hat{S}_z\right) =$$

$$\frac{i\omega_0}{\hbar}cos(\omega t)\hat{S}_z exp\left(\frac{i\omega_0}{\hbar\omega}sin(\omega t)\hat{S}_z\right)$$

The time evolution operator $\hat{U}(t)$:

$$\hat{U}(t) = exp\left(\frac{i\omega_0}{\hbar\omega}sin(\omega t)\hat{S}_z\right)$$

We remember:

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$$

The complete time evolution operator $\hat{U}(t)$:

$$\hat{U}(t) = exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\hat{\sigma}_z\right)$$

Rotation

We calculate the rotation by calculating the angles θ and φ.

The spin is precessing in the $x - y$ plane, θ is $\frac{\pi}{2}$ (90° degree from the z-axis).

We apply the unitary time evolution operator to the initial state (spin in x-direction).

The time evolution operator:

$$\hat{U}(t) = exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\hat{\sigma}_z\right) =$$

$$\hat{U}(t) = exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) =$$

$$\begin{pmatrix} exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right) & 0 \\ 0 & exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right)\end{pmatrix}$$

We apply the matrix to the state $|\psi, 0\rangle$:

$$\hat{U}(t)|\psi, 0\rangle =$$

$$\begin{pmatrix} exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right) & 0 \\ 0 & exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right) \end{pmatrix} \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right) =$$

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix} exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right) \end{pmatrix}\right) =$$

$$\frac{1}{\sqrt{2}}exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}0\\1\end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right)\left(\begin{pmatrix}1\\0\end{pmatrix} + exp\left(-i\frac{\omega_0}{\omega}sin(\omega t)\right)\begin{pmatrix}0\\1\end{pmatrix}\right)$$

Result:

$$\varphi(t) = -\frac{\omega_0}{\omega}sin(\omega t)$$

Note: $\frac{1}{\sqrt{2}}exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right)$ is a global outer phase that does not disturb the direction of the spin.

As in the example above we ask: What is the time-dependent probability (the expectation value) to find the spin in the opposite direction?

We calculate the expectation value for the transition:

$$\left\langle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \middle| \hat{U}(t) \middle| \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \right\rangle$$

The right side is already done:

$$\hat{U}(t)|\psi,0\rangle = \frac{1}{\sqrt{2}}exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}0\\1\end{pmatrix}$$

We calculate:

$$\left\langle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \middle| \frac{1}{\sqrt{2}}exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}0\\1\end{pmatrix} \right\rangle =$$

$$\frac{1}{2}\left\langle \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} \middle| exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right)\begin{pmatrix}0\\1\end{pmatrix} \right\rangle =$$

$$\frac{1}{2}\left(exp\left(\frac{i\omega_0}{2\omega}sin(\omega t)\right) - exp\left(-\frac{i\omega_0}{2\omega}sin(\omega t)\right)\right) =$$

$$i \cdot sin\left(\frac{\omega_0}{2\omega}sin(\omega t)\right)$$

Note: for complex numbers hold:

$$z - \bar{z} = 2 \cdot \ img(z)$$

The probability is the square of the probability amplitude, so we calculate:

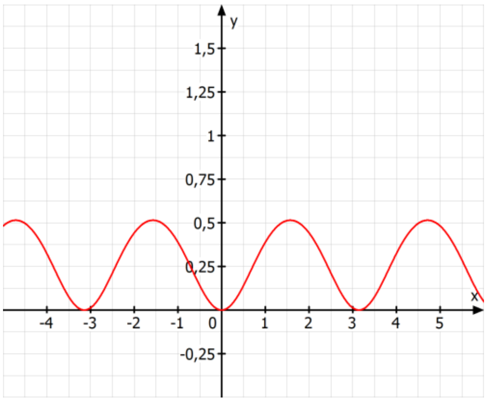
$$P = -i \cdot sin\left(\frac{\omega_0}{2\omega} sin(\omega t)\right) \cdot i \cdot sin\left(\frac{\omega_0}{2\omega} sin(\omega t)\right) =$$

$$sin^2\left(\frac{\omega_0}{2\omega} sin(\omega t)\right)$$

Physical interpretation

This allows physical interpretation.

- $sin(\omega t)$ does not change the maximum amplitude of the probability, only its frequency.
- If $\frac{\omega_0}{2\omega} < \frac{\pi}{2}$ we have no probability for P of 100%.



Note: the function shown is $sin^2(0.8 \cdot sin(x))$

With this we get the maximum frequency ω that allows a swing of 180°:

$$\varphi(t) = -\frac{\omega_0}{\omega} sin(\omega t) = \pi$$

$$\frac{\omega_0}{\omega} = \pi$$

$$\omega = \frac{\omega_0}{\pi}$$

Approach via differential equations

We calculate the unitary time development operator $\hat{U}(t)$ by use of a differential ansatz. As before we have the Hamiltonian:

$$\hat{H} = -\omega_0 \cdot cos(\omega t)\overrightarrow{S_z}$$

We begin:

$$\frac{d\hat{U}(t)}{dt} = \frac{-i}{\hbar}\hat{H}(t)\hat{U}(t) = \frac{i\omega_0}{2} cos(\omega t)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\hat{U}(t)$$

This is a differential equation. We get:

$$\hat{U}(t) = exp\left(\frac{i\omega_0}{\hbar\omega} sin(\omega t)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\right)$$

We try:

$$\frac{d\widehat{U}(t)}{dt} = \frac{d}{dt} exp\left(\frac{i\omega_0}{\hbar\omega} sin(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) =$$

$$exp\left(\frac{i\omega_0}{\hbar\omega} sin(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot \left(\frac{i\omega_0}{\hbar\omega} \omega cos(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) =$$

$$exp\left(\frac{i\omega_0}{\hbar\omega} sin(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot \left(\frac{i\omega_0}{\hbar} cos(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) =$$

$$\widehat{U}(t) \cdot \left(\frac{i\omega_0}{\hbar} cos(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

The Schrödinger equation:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{-i}{\hbar}\widehat{H}|\psi(t)\rangle = \frac{i}{2}\omega_0 cos(\omega t)\hat{\sigma}_z|\psi(t)\rangle =;$$

We set:

$$|\psi(t)\rangle := \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

We get from the Schrödinger equation:

$$\begin{pmatrix} \frac{\partial a(t)}{\partial t} \\ \frac{\partial b(t)}{\partial t} \end{pmatrix} = \frac{i}{2}\omega_0 cos(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{i}{2}\omega_0 cos(\omega t) \begin{pmatrix} a(t) \\ -b(t) \end{pmatrix}$$

From this we have:

$$\frac{\partial a(t)}{\partial t} = \frac{i}{2}\omega_0 cos(\omega t) a(t)$$

$$\frac{\partial b(t)}{\partial t} = -\frac{i}{2}\omega_0 cos(\omega t) b(t)$$

These are differential equations with solutions:

$$a(t) = \exp\left(\frac{i}{2}\frac{\omega_0}{\omega} sin(\omega t)\right) + c$$

$$b(t) = \exp\left(-\frac{i}{2}\frac{\omega_0}{\omega} sin(\omega t)\right) + d$$

$a(0) = b(0) \approx 1$ except for the normalization factor, so we set c and d zero and get the unitary operator that generates time evolution:

$$\widehat{U}(t) = \begin{pmatrix} \exp\left(\frac{i}{2}\frac{\omega_0}{\omega} sin(\omega t)\right) & 0 \\ 0 & \exp\left(-\frac{i}{2}\frac{\omega_0}{\omega} sin(\omega t)\right) \end{pmatrix}$$

Note: for diagonal matrices D holds:

$$e^{\begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & n \end{pmatrix}} = \begin{pmatrix} e^a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^n \end{pmatrix}$$

We write:

$$\hat{U}(t) = exp\left(\frac{i\omega_0}{2\omega} sin(\omega t)\hat{\sigma}_z\right)$$

The same result we got with the power series expansion.

Rotating magnetic field

We change from a static magnetic field to a rotating one. The field $\vec{B}(t)$ is the sum of B_0 constant in z-direction and B_1 rotating in the $x - y$ plane:

$$\vec{B}(t) = B_0 \cdot z + B_1\big(x \cdot cos(\omega t) - y \cdot sin(\omega t)\big)$$

Note: B_1 is weak, $B_0 \gg B_1 > 0$.

Note: at time $t = 0$, B_1 points into x-direction.

Note: the rotation is clockwise.

The time-dependent Hamiltonian of the system:

$$\hat{H}(t) = -\gamma \vec{B}(t) \cdot \vec{S}$$

Note: $\vec{S}$ is the spin operator:

$$\hat{S} \coloneqq \hat{S}_x + \hat{S}_y + \hat{S}_z$$

We get:

$$\hat{H}(t) = -\gamma\left(B_0 \cdot \hat{S}_z + B_1\left(\hat{S}_x \cdot cos(\omega t) - \hat{S}_y \cdot sin(\omega t)\right)\right)$$

Note: the Hamiltonians at different time do not commute. We split the problem by

- first working with a rotating frame Ψ_R that brings us back to a constant field
- generalizing the solution $\Psi_R(t) \to \Psi(t)$

Zero Hamiltonian

Let $\vec{B}(t) = 0$. In this case the Hamiltonian is zero, $\hat{H}(t) = 0$.

If we now rotate the frame, the (frame-)Hamiltonian cannot be zero because the spin states are rotating around the z-axis.

The Hamiltonian for a "rotating" spin:

$$\hat{H}_U = \omega \cdot \hat{S}_z$$

Note: we name the Hamiltonian $\hat{H}_U$ because it is constructed by the rotating frame. The unitary operator constituting this frame:

$$\hat{U}(t) = e^{-\frac{i\omega \hat{S}_z t}{\hbar}} = e^{-\frac{i\hat{H}_U t}{\hbar}}$$

Nonzero Hamiltonian

Let $B(t) \neq 0$. In this case the Hamiltonian is not zero, $\hat{H}(t) \neq 0$.

The rotating-frame state:

$$|\Psi_R, t\rangle := \hat{U}(t)|\Psi, t\rangle = \hat{U}(t)\hat{U}_S(t)|\Psi, 0\rangle$$

Note: $|\Psi, t\rangle = \hat{U}_S(t)|\Psi, 0\rangle$

For operators $\hat{U}(t) = e^{-\frac{i\hat{H}_U t}{\hbar}}$ we have:

$$\frac{\partial}{\partial t}\hat{U} = -\frac{i\hat{H}_U}{\hbar}\hat{U}$$

We calculate the Hamiltonian:

$$i\hbar\left(\frac{\partial}{\partial t}\hat{U}\right)\hat{U}^\dagger = i\hbar\left(-\frac{i\hat{H}_U}{\hbar}\hat{U}\right)\hat{U}^\dagger = \hat{H}_U$$

The Hamiltonian for the rotating frame:

$$\hat{H}_R = i\hbar\left(\frac{\partial}{\partial t}\hat{U}(t)\hat{U}_S(t)\right)\hat{U}_S{}^\dagger(t)\hat{U}^\dagger(t) =$$

$$i\hbar\left(\left(\frac{\partial}{\partial t}\hat{U}(t)\right)\hat{U}_S(t) + \hat{U}(t)\left(\frac{\partial}{\partial t}\hat{U}_S(t)\right)\right)\hat{U}_S{}^\dagger(t)\hat{U}^\dagger(t) =$$

$$i\hbar\left(\left(-\frac{i\hat{H}_U}{\hbar}\hat{U}(t)\right)\hat{U}_S(t) + \hat{U}(t)\left(-\frac{i\hat{H}_S}{\hbar}\hat{U}_S(t)\right)\right)\hat{U}_S{}^\dagger(t)\hat{U}^\dagger(t) =$$

$$\left(\left(\hat{H}_U\hat{U}(t)\right)\hat{U}_S(t) + \hat{U}(t)\left(\hat{H}_S\hat{U}_S(t)\right)\right)\hat{U}_S{}^\dagger(t)\hat{U}^\dagger(t) =$$

$$= \hat{H}_U\hat{U}(t)\hat{U}^\dagger(t) + \hat{U}(t)\hat{H}_S\hat{U}^\dagger(t)$$

$$= \hat{H}_U(t) + \hat{U}(t)\hat{H}_S\hat{U}^\dagger(t)$$

Note: in the case of no magnetic field, $\hat{H}_S = 0$, we get the Hamiltonian for the rotation, $\hat{H}_U$.

We resolve the composed Hamiltonian:

$$\hat{H}_R = \hat{H}_U(t) + \hat{U}(t)\hat{H}_S\hat{U}^\dagger(t) =$$

$$\omega \cdot \hat{S}_z - e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\gamma\left(B_0 \cdot \hat{S}_z + B_1\left(\hat{S}_x \cdot cos(\omega t) - \hat{S}_y \cdot sin(\omega t)\right)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}} =$$

$$\omega \cdot \hat{S}_z - e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\gamma \cdot B_0 \cdot \hat{S}_z \cdot e^{\frac{i\omega\hat{S}_z t}{\hbar}} - e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\gamma B_1\left(\hat{S}_x \cdot cos(\omega t) - \hat{S}_y \cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}} =;$$

Note: $e^{-\frac{i\omega\hat{S}_z t}{\hbar}}$ is commuting with $\hat{S}_z$

$$\omega \cdot \hat{S}_z - \gamma \cdot B_0 \cdot \hat{S}_z - e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\gamma B_1\left(\hat{S}_x \cdot cos(\omega t) - \hat{S}_y \cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}} =$$

$$(-\gamma B_0 + \omega) \cdot \hat{S}_z - \gamma B_1 \cdot \underbrace{e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\hat{S}_x \cdot cos(\omega t) - \hat{S}_y \cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}} =;$$

We concentrate on the rotation:

$$e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}$$

There is a chance that the rotation has no time dependence. In this case the time-derivative should give zero. We check this:

$$\frac{\partial}{\partial t}\left(e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}\right)=$$

$$\begin{aligned}-\frac{i\omega\hat{S}_z}{\hbar}e^{-\frac{i\omega\hat{S}_z t}{\hbar}}&\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}\\&+e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\frac{\partial}{\partial t}\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}\\&+e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)\frac{i\omega\hat{S}_z}{\hbar}e^{\frac{i\omega\hat{S}_z t}{\hbar}}=;\end{aligned}$$

Note: $\hat{S}_z$ commutes with $e^{-\frac{i\omega\hat{S}_z t}{\hbar}}$.

$$\begin{aligned}e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\Bigg(-\frac{i\omega\hat{S}_z}{\hbar}&\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)+\left(\frac{\partial}{\partial t}\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)\right)\\&+\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)\frac{i\omega\hat{S}_z}{\hbar}\Bigg)e^{\frac{i\omega\hat{S}_z t}{\hbar}}=\end{aligned}$$

$$e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(-\frac{i\omega}{\hbar}\left[\hat{S}_z,\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right]+\left(\frac{\partial}{\partial t}\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)\right)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}=$$

$$e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(-\frac{i\omega}{\hbar}\left[\hat{S}_z,\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)\right]+\left(-\hat{S}_x\cdot sin(\omega t)\cdot\omega-\hat{S}_y\cdot cos(\omega t)\cdot\omega\right)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}=$$

$$\begin{aligned}e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\Bigg(-\frac{i\omega}{\hbar}&\left(\left[\hat{S}_z,\hat{S}_x\right]cos(\omega t)-\left[\hat{S}_z,\hat{S}_y\right]sin(\omega t)\right)\\&+\left(-\hat{S}_x\cdot sin(\omega t)\cdot\omega-\hat{S}_y\cdot cos(\omega t)\cdot\omega\right)\Bigg)e^{\frac{i\omega\hat{S}_z t}{\hbar}}=;\end{aligned}$$

We remember the commutator relations:

$$\left[\hat{S}_x,\hat{S}_y\right]=i\hbar\hat{S}_z;\ \left[\hat{S}_y,\hat{S}_z\right]=i\hbar\hat{S}_x;\ \left[\hat{S}_z,\hat{S}_x\right]=i\hbar\hat{S}_y$$

$$e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(-\frac{i\omega}{\hbar}\left(i\hbar\hat{S}_y cos(\omega t)+i\hbar\hat{S}_x sin(\omega t)\right)+\left(-\hat{S}_x\cdot sin(\omega t)\cdot\omega-\hat{S}_y\cdot cos(\omega t)\cdot\omega\right)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}=$$

$$e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\omega\left(\hat{S}_y cos(\omega t)+\hat{S}_x sin(\omega t)\right)+\left(-\hat{S}_x\cdot sin(\omega t)\cdot\omega-\hat{S}_y\cdot cos(\omega t)\cdot\omega\right)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}=0$$

As we have no time dependence in:

$$\gamma B_1\cdot e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}$$

… we have no time dependence in the rotating frame Hamiltonian $\hat{H}_R$:

$$(-\gamma B_0+\omega)\cdot\hat{S}_z-\gamma B_1\cdot e^{-\frac{i\omega\hat{S}_z t}{\hbar}}\left(\hat{S}_x\cdot cos(\omega t)-\hat{S}_y\cdot sin(\omega t)\right)e^{\frac{i\omega\hat{S}_z t}{\hbar}}$$

We are allowed to fix the rotation expression at any time. The simplest one is $t = 0$:

$$(-\gamma B_0 + \omega) \cdot \hat{S}_z - \gamma B_1 \cdot e^{-\frac{i\omega \hat{S}_z t=0}{\hbar}} \left(\hat{S}_x \cdot cos(\omega t = 0) - \hat{S}_y \cdot sin(\omega t = 0)\right) e^{\frac{i\omega \hat{S}_z t=0}{\hbar}} \rightarrow$$

$$(-\gamma B_0 + \omega) \cdot \hat{S}_z - \gamma B_1 \hat{S}_x =;$$

To preserve both magnetic components B_0 and B_1, we do some tricky transformations:

$$\left(-\gamma B_0 + \frac{\gamma B_0 \omega}{\gamma B_0}\right) \cdot \hat{S}_z - \gamma B_1 \hat{S}_x =$$

$$\gamma B_0 \left(-1 + \frac{\omega}{\gamma B_0}\right) \cdot \hat{S}_z - \gamma B_1 \hat{S}_x =;$$

We remember the Larmor frequency ω_0 associated with a constant magnetic field:

$$\omega_0 = \gamma B_0$$

$$\gamma B_0 \left(-1 + \frac{\omega}{\omega_0}\right) \cdot \hat{S}_z - \gamma B_1 \hat{S}_x =$$

$$-\gamma B_0 \left(1 - \frac{\omega}{\omega_0}\right) \cdot \hat{S}_z - \gamma B_1 \hat{S}_x =$$

$$-\gamma \left(B_0 \left(1 - \frac{\omega}{\omega_0}\right) \cdot \hat{S}_z + B_1 \hat{S}_x\right)$$

Result:

$$\hat{H}_R = -\gamma \left(B_0 \left(1 - \frac{\omega}{\omega_0}\right) \cdot \hat{S}_z + B_1 \hat{S}_x\right)$$

The Hamiltonian $\hat{H}_R$ thus can be expressed with the scalar product:

$$\hat{H}_R = -\gamma \hat{B}_R \cdot \hat{S}$$

$$\hat{B}_R = B_1 x + B_0 \left(1 - \frac{\omega}{\omega_0}\right) z$$

The state $|\Psi, t\rangle$

Back to our rotating-frame state. We got the rotating frame state $|\Psi_R, t\rangle$ from the state $|\Psi, t\rangle$ by applying the unitary rotation operator $\hat{U}(t) = e^{-\frac{i\omega \hat{S}_z t}{\hbar}}$:

$$|\Psi_R, t\rangle \coloneqq \hat{U}(t)|\Psi, t\rangle$$

We can extract $|\Psi, t\rangle$ by applying the inverse operator $\hat{U}^{-1}(t)$:

$$|\Psi, t\rangle = \hat{U}^{-1}(t)|\Psi_R, t\rangle = e^{\frac{i\omega \hat{S}_z t}{\hbar}}|\Psi_R, t\rangle$$

We know that $\hat{H}_R$ is time-independent and calculate the time-evolution of $|\Psi_R, t\rangle$:

$$|\Psi_R, t\rangle = e^{\left(-i\frac{\hat{H}_R}{\hbar}t\right)}|\Psi_R, 0\rangle$$

At time $t = 0$ the state $|\Psi_R, 0\rangle$ is equal to $|\Psi, 0\rangle$:

$$|\Psi, t\rangle = e^{\frac{i\omega \hat{S}_z t}{\hbar}} e^{\left(-i\frac{\hat{H}_R}{\hbar}t\right)}|\Psi_R, 0\rangle =$$

$$e^{\frac{i\omega\hat{S}_z t}{\hbar}} e^{\left(i\frac{\gamma\hat{B}_R\cdot\hat{S}}{\hbar}t\right)}|\Psi,0\rangle$$

Physical interpretation

Note: the following discussion is valid for $B_1 \ll B_0$. ω is the rotating speed of B_1, ω_0 is the Larmor frequency associated with B_0.

If the magnetic field B_1 is rotating slower than the leading field B_0, $\omega \ll \omega_0$ we have nearly no effect:

$$\hat{B}_R = B_1 x + B_0\left(1 - \frac{\omega}{\omega_0}\right)z \to B_1 x + B_0 z$$

This resembles a slightly disturbed field B_0 pointing to a slightly tilted z-axis:

$$|\Psi,t\rangle = e^{\frac{i\omega\hat{S}_z t}{\hbar}} e^{\left(i\frac{\gamma\hat{B}_R\cdot\hat{S}}{\hbar}t\right)}|\Psi,0\rangle$$

The spin is precessing round this tilted z-axis with a rate about ω_0:

$$e^{\left(i\frac{\gamma\hat{B}_R\cdot\hat{S}}{\hbar}t\right)} \to e^{\left(i\frac{\gamma B_0\cdot S_z}{\hbar}t\right)}$$

$e^{\frac{i\omega\hat{S}_z t}{\hbar}}$ adds a rotation round the z-axis with smaller angular velocity ω. The original precession is slightly disturbed.

The interesting case comes up if we set $\omega = \omega_0$. Then we have:

$$\hat{B}_R = B_1 x + B_0\left(1 - \frac{\omega_0}{\omega_0}\right)z = B_1 x$$

The strong magnetic field B_0 seems to be cancelled out by frequency.

We get:

$$|\Psi,t\rangle = e^{\frac{i\omega\hat{S}_z t}{\hbar}} e^{\left(i\frac{\gamma B_1 x\cdot\hat{S}}{\hbar}t\right)}|\Psi,0\rangle = e^{\frac{i\omega\hat{S}_z t}{\hbar}} e^{\left(i\frac{\gamma B_1 S_x}{\hbar}t\right)}$$

In detail:

$e^{\left(i\frac{\gamma B_1 S_x}{\hbar}t\right)}$ describes the spin precessing round the x-axis. The spin, initially pointing towards the z-axis will rotate to the y-axis with angular velocity $\gamma B_1 \coloneqq \omega_1$. We have $B_1 \ll B_0$, so:

$$\omega_1 \ll \omega_0$$

The spin will rotate round the x-axis and reach the y-axis after time T_y:

$$T_y = \frac{\pi}{2\gamma B_1}$$

We still have the rotation round the z-axis. This rotation is faster than the rotation round the x-axis.

The tip of the spin performs a spiral motion on the surface of a hemisphere till it reaches the $x-y$-plane.