This paper deals with expectation values in a square well. It is based on Problem 5, Chapter 3, of Hans Ohanian's “Principles of Quantum Mechanics".

Note: Without a CAS the calculations will not be easy to handle.

A good place for the integrals is https://www.integralrechner.de/

Hope I can help you learning quantum mechanics.

A particle of mass m moves in an infinite square well with $x \in [0, a]$.

We know the eigenstates of the infinite square well:

$$\Psi_n(x) = \sqrt{\frac{2}{a}} sin\left(\frac{n\pi x}{a}\right), n = 1,2,3, \ldots$$

We know the associated energies:

$$E_n = \frac{n^2 \hbar^2 \pi^2}{2ma^2}$$

We use the wave function:

$$\Psi(x, 0) = \sqrt{\frac{1}{3}} \sqrt{\frac{2}{a}} sin\left(\frac{2\pi x}{a}\right) + \sqrt{\frac{2}{3}} \sqrt{\frac{2}{a}} sin\left(\frac{3\pi x}{a}\right)$$

This is not an energy eigenstate (the function is not periodic with respect to a …), but it is a superposition of the eigenstates:

$$\Psi_2(x) = \sqrt{\frac{2}{a}} sin\left(\frac{2\pi x}{a}\right)$$

$$\Psi_3(x) = \sqrt{\frac{2}{a}} sin\left(\frac{3\pi x}{a}\right)$$

$$\Psi(x, 0) = \sqrt{\frac{1}{3}} \Psi_2(x) + \sqrt{\frac{2}{3}} \Psi_3(x)$$

We use the time evolution of energy eigenstates:

$$\Psi(x, t) = \sqrt{\frac{1}{3}} \Psi_2(x) exp\left(-i \frac{E_2}{\hbar} t\right) + \sqrt{\frac{2}{3}} \Psi_3(x) xp\left(-i \frac{E_3}{\hbar} t\right)$$

We resolve the energies E_2 and E_3:

$$E_2 = \frac{4\hbar^2\pi^2}{ma^2}, E_3 = \frac{9\hbar^2\pi^2}{2ma^2}$$

We will need the difference $E_2 - E_3$:

$$\frac{4\hbar^2\pi^2}{ma^2} - \frac{9\hbar^2\pi^2}{2ma^2} = -\frac{5\hbar^2\pi^2}{2ma^2}$$

We calculate:

$$\Psi(x, t) = \sqrt{\frac{1}{3}} \Psi_2(x) exp\left(-i \frac{E_2}{\hbar} t\right) + \sqrt{\frac{2}{3}} \Psi_3(x) exp\left(-i \frac{E_3}{\hbar} t\right) =$$

$$\sqrt{\frac{1}{3}} \sqrt{\frac{2}{a}} sin\left(\frac{2\pi x}{a}\right) exp\left(-i \frac{E_2}{\hbar} t\right) + \sqrt{\frac{2}{3}} \sqrt{\frac{2}{a}} sin\left(\frac{3\pi x}{a}\right) exp\left(-i \frac{E_3}{\hbar} t\right) =$$

$$\sqrt{\frac{2}{3a}}sin\left(\frac{2\pi x}{a}\right)exp\left(-i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{4}{3a}}sin\left(\frac{3\pi x}{a}\right)exp\left(-i\frac{E_3}{\hbar}t\right)$$

We calculate the expectation value for position $\langle x\rangle$:

$$\langle x\rangle=\int_0^a\Psi^*(x,t)\cdot\hat{x}\cdot\Psi(x,t)dx=;$$

We calculate the interior of the integral, remembering that $\Psi(x,t)$ is a real function and the operator $\hat{x}$ becomes the multiplication with x:

$$\Psi^*(x,t)\cdot\hat{x}\cdot\Psi(x,t)=$$

$$x\cdot\left(\sqrt{\frac{1}{3}}\Psi_2(x)exp\left(i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{2}{3}}\Psi_3(x)exp\left(i\frac{E_3}{\hbar}t\right)\right)$$

$$\cdot\left(\sqrt{\frac{1}{3}}\Psi_2(x)exp\left(-i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{2}{3}}\Psi_3(x)exp\left(-i\frac{E_3}{\hbar}t\right)\right)=$$

$$x\cdot\left(\frac{1}{3}\left(\Psi_2(x)\right)^2+\sqrt{\frac{2}{9}}\Psi_2(x)\Psi_3(x)exp\left(i\frac{E_2-E_3}{\hbar}t\right)+\sqrt{\frac{2}{9}}\Psi_3(x)\Psi_2(x)exp\left(i\frac{E_3-E_2}{\hbar}t\right)\right.$$

$$\left.+\frac{2}{3}\left(\Psi_3(x)\right)^2\right)=$$

$$x\cdot\left(\frac{2}{3a}\left(sin\left(\frac{2\pi x}{a}\right)\right)^2+\sqrt{\frac{8}{9a^2}}sin\left(\frac{2\pi x}{a}\right)sin\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_2-E_3}{\hbar}t\right)\right.$$

$$\left.+\sqrt{\frac{8}{9a^2}}sin\left(\frac{3\pi x}{a}\right)sin\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_3-E_2}{\hbar}t\right)+\frac{4}{3a}\left(sin\left(\frac{3\pi x}{a}\right)\right)^2\right)=$$

$$x\cdot\left(\frac{2}{3a}\left(sin\left(\frac{2\pi x}{a}\right)\right)^2+\sqrt{\frac{8}{9a^2}}sin\left(\frac{2\pi x}{a}\right)sin\left(\frac{3\pi x}{a}\right)\left(exp\left(i\frac{E_2-E_3}{\hbar}t\right)+exp\left(i\frac{E_3-E_2}{\hbar}t\right)\right)\right.$$

$$\left.+\frac{4}{3a}\left(sin\left(\frac{3\pi x}{a}\right)\right)^2\right)=$$

$$x\cdot\left(\frac{2}{3a}\left(sin\left(\frac{2\pi x}{a}\right)\right)^2+2\sqrt{\frac{8}{9a^2}}sin\left(\frac{2\pi x}{a}\right)sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{E_2-E_3}{\hbar}t\right)+\frac{4}{3a}\left(sin\left(\frac{3\pi x}{a}\right)\right)^2\right)=$$

$$x\cdot\frac{2}{3a}\left(\left(sin\left(\frac{2\pi x}{a}\right)\right)^2+\sqrt{8}sin\left(\frac{2\pi x}{a}\right)sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{E_2-E_3}{\hbar}t\right)+2\left(sin\left(\frac{3\pi x}{a}\right)\right)^2\right)$$

We look at Wikipedia:

$$sin\left(\frac{2\pi x}{a}\right)sin\left(\frac{3\pi x}{a}\right)=\frac{1}{2}cos\left(-\frac{\pi x}{a}\right)-\frac{1}{2}cos\left(\frac{5\pi x}{a}\right)$$

We build the complete integral:

$$\frac{2}{3a}\int_0^a x\cdot\left(sin\left(\frac{2\pi x}{a}\right)\right)^2 dx+\frac{2}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)\int_0^a x\cdot 2\sqrt{2}\frac{1}{2}cos\left(-\frac{\pi x}{a}\right)dx$$
$$-\frac{2}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)\int_0^a x\cdot 2\sqrt{2}\frac{1}{2}cos\left(\frac{5\pi x}{a}\right)dx+\frac{2}{3a}\int_0^a 2x\left(sin\left(\frac{3\pi x}{a}\right)\right)^2 dx=$$

$$\frac{2}{3a}\int_0^a x\cdot\left(sin\left(\frac{2\pi x}{a}\right)\right)^2 dx+\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)\int_0^a x\cdot cos\left(-\frac{\pi x}{a}\right)dx$$
$$-\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)\int_0^a x\cdot cos\left(\frac{5\pi x}{a}\right)dx+\frac{4}{3a}\int_0^a x\left(sin\left(\frac{3\pi x}{a}\right)\right)^2 dx=;$$

We use https://www.integralrechner.de/

1 $$\frac{2}{3a}\int_0^a x\cdot\left(sin\left(\frac{2\pi x}{a}\right)\right)^2 dx=\frac{a^2}{4}\frac{2}{3a}=\frac{1}{6}a$$

2 $$\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)\int_0^a x\cdot cos\left(-\frac{\pi x}{a}\right)dx=-\frac{2a^2}{\pi^2}\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)$$

3 $$-\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)\int_0^a x\cdot cos\left(\frac{5\pi x}{a}\right)dx=\frac{2a^2}{25\pi^2}\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)$$

4 $$\frac{4}{3a}\int_0^a x\left(sin\left(\frac{3\pi x}{a}\right)\right)^2 dx=\frac{a^2}{4}\frac{4}{3a}=\frac{1}{3}a$$

We build the middle sum 2 + 3:

$$\frac{2a^2}{25\pi^2}\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)-\frac{2a^2}{\pi^2}\frac{2\sqrt{2}}{3a}cos\left(\frac{E_2-E_3}{\hbar}t\right)=$$

$$\left(\frac{2a^2}{25\pi^2}\frac{2\sqrt{2}}{3a}-\frac{2a^2}{\pi^2}\frac{2\sqrt{2}}{3a}\right)cos\left(\frac{E_2-E_3}{\hbar}t\right)=$$

$$\frac{2a^2 2\sqrt{2}}{\pi^2 3a}\left(\frac{1}{25}-\frac{25}{25}\right)cos\left(\frac{E_2-E_3}{\hbar}t\right)=$$

$$\frac{2a^2 2\sqrt{2}}{\pi^2 3a}\left(-\frac{24}{25}\right)cos\left(\frac{E_2-E_3}{\hbar}t\right)=$$

$$-\frac{96a\sqrt{2}}{75\pi^2}cos\left(\frac{E_2-E_3}{\hbar}t\right)=$$

$$-\frac{32a\sqrt{2}}{25\pi^2}cos\left(\frac{E_2-E_3}{\hbar}t\right)$$

Result:

$$\langle x\rangle=\frac{1}{2}a-\frac{32a\sqrt{2}}{25\pi^2}cos\left(\frac{E_2-E_3}{\hbar}t\right)$$

We resolve the energy difference E_2-E_3:

$$E_2-E_3=-\frac{5\hbar^2\pi^2}{2ma^2}$$

We get:

$$\langle x\rangle = a\left(\frac{1}{2}-\frac{32\sqrt{2}}{25\pi^2}cos\left(-\frac{5\hbar\pi^2}{2ma^2}t\right)\right)$$

Quickcheck:

$$\frac{32a\sqrt{2}}{25\pi^2}\cong 0.18 \Longrightarrow \langle x\rangle \in [0,a]$$

We calculate the expectation value for momentum $\langle p\rangle$:

$$\langle p\rangle = \int_0^a \Psi^*(x,t)\cdot\hat{p}\cdot\Psi(x,t)dx$$

The momentum operator in position representation:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

The wave function:

$$\Psi(x,t) = \sqrt{\frac{2}{3a}}sin\left(\frac{2\pi x}{a}\right)exp\left(-i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{4}{3a}}sin\left(\frac{3\pi x}{a}\right)exp\left(-i\frac{E_3}{\hbar}t\right)$$

The expectation value for momentum $\langle p\rangle$:

$$\langle p\rangle = -i\hbar\int_0^a \sqrt{\frac{2}{3a}}sin\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{4}{3a}}sin\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_3}{\hbar}t\right)$$
$$\cdot\frac{\partial}{\partial x}\left(\sqrt{\frac{2}{3a}}sin\left(\frac{2\pi x}{a}\right)exp\left(-i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{4}{3a}}sin\left(\frac{3\pi x}{a}\right)exp\left(-i\frac{E_3}{\hbar}t\right)\right)dx =$$

$$-i\hbar\int_0^a \sqrt{\frac{2}{3a}}sin\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{4}{3a}}sin\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_3}{\hbar}t\right)$$
$$\cdot\left(\sqrt{\frac{2}{3a}}\frac{2\pi}{a}cos\left(\frac{2\pi x}{a}\right)exp\left(-i\frac{E_2}{\hbar}t\right)+\sqrt{\frac{4}{3a}}\frac{3\pi}{a}cos\left(\frac{3\pi x}{a}\right)exp\left(-i\frac{E_3}{\hbar}t\right)\right)dx =$$

$$-i\hbar\int_0^a \left(\left(\sqrt{\frac{2}{3a}}sin\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_2}{\hbar}t\right)\right)\left(\sqrt{\frac{2}{3a}}\frac{2\pi}{a}cos\left(\frac{2\pi x}{a}\right)exp\left(-i\frac{E_2}{\hbar}t\right)\right)\right.$$
$$+\left(\sqrt{\frac{2}{3a}}sin\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_2}{\hbar}t\right)\right)\left(\sqrt{\frac{4}{3a}}\frac{3\pi}{a}cos\left(\frac{3\pi x}{a}\right)exp\left(-i\frac{E_3}{\hbar}t\right)\right)$$
$$+\left(\sqrt{\frac{4}{3a}}sin\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_3}{\hbar}t\right)\right)\left(\sqrt{\frac{2}{3a}}\frac{2\pi}{a}cos\left(\frac{2\pi x}{a}\right)exp\left(-i\frac{E_2}{\hbar}t\right)\right)$$
$$\left.+\left(\sqrt{\frac{4}{3a}}sin\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_3}{\hbar}t\right)\right)\left(\sqrt{\frac{4}{3a}}\frac{3\pi}{a}cos\left(\frac{3\pi x}{a}\right)exp\left(-i\frac{E_3}{\hbar}t\right)\right)dx=\right.$$

$$-i\hbar\int_0^a \left(\frac{2}{3a}\frac{2\pi}{a}sin\left(\frac{2\pi x}{a}\right)\right)\left(cos\left(\frac{2\pi x}{a}\right)\right)+\left(\sqrt{\frac{2}{3a}}\sqrt{\frac{4}{3a}}\frac{3\pi}{a}sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_2-E_3}{\hbar}t\right)\right)$$
$$+\left(\sqrt{\frac{4}{3a}}\sqrt{\frac{2}{3a}}\frac{2\pi}{a}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_3-E_2}{\hbar}t\right)\right)$$
$$+\left(\sqrt{\frac{4}{3a}}\sqrt{\frac{4}{3a}}\frac{3\pi}{a}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)\right)dx=$$

$$-i\hbar\int_0^a \left(\frac{4\pi}{3a^2}sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)\right)+\left(\sqrt{\frac{8}{9a^2}}\frac{3\pi}{a}sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_2-E_3}{\hbar}t\right)\right)$$
$$+\left(\sqrt{\frac{8}{9a^2}}\frac{2\pi}{a}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_3-E_2}{\hbar}t\right)\right)$$
$$+\left(\frac{12\pi}{3a^2}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)\right)dx=$$

$$-i\hbar\int_0^a \left(\frac{4\pi}{3a^2}sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)\right)+\left(\sqrt{\frac{8}{9a^2}}\frac{3\pi}{a}sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_2-E_3}{\hbar}t\right)\right)$$
$$+\left(\sqrt{\frac{8}{9a^2}}\frac{2\pi}{a}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_3-E_2}{\hbar}t\right)\right)$$
$$+\left(\frac{12\pi}{3a^2}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)\right)dx=$$

$$-i\hbar\int_0^a\left(\frac{4\pi}{3a^2}sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)\right)+\left(\sqrt{2}\frac{6\pi}{3a^2}sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_2-E_3}{\hbar}t\right)\right)$$
$$+\left(\sqrt{2}\frac{4\pi}{3a^2}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_3-E_2}{\hbar}t\right)\right)$$
$$+\left(\frac{12\pi}{3a^2}sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)\right)dx =$$

$$-i\hbar\frac{2\pi}{3a^2}\int_0^a\left(2\cdot sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)\right)+\left(\sqrt{2}\cdot 3sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)exp\left(i\frac{E_2-E_3}{\hbar}t\right)\right)$$
$$+\left(\sqrt{2}\cdot 2sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)exp\left(i\frac{E_3-E_2}{\hbar}t\right)\right)+\left(6sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)\right)dx =$$

$$-i\hbar\frac{2\pi}{3a^2}\int_0^a 2\cdot sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)dx + \qquad 1$$
$$-i\hbar\frac{2\pi}{3a^2}exp\left(i\frac{E_2-E_3}{\hbar}t\right)\int_0^a \sqrt{2}\cdot 3sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)dx + \qquad 2$$
$$-i\hbar\frac{2\pi}{3a^2}exp\left(i\frac{E_3-E_2}{\hbar}t\right)\int_0^a \sqrt{2}\cdot 2sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)dx + \qquad 3$$
$$-i\hbar\frac{2\pi}{3a^2}\int_0^a 6\cdot sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)dx =; \qquad 4$$

The integrals:

$$1 \qquad -i\hbar\frac{2\pi}{3a^2}\int_0^a 2\cdot sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)dx = 0$$

$$2 \qquad -i\hbar\frac{2\pi}{3a^2}exp\left(i\frac{E_2-E_3}{\hbar}t\right)\int_0^a \sqrt{2}\cdot 3sin\left(\frac{2\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)dx =$$
$$-i\hbar\frac{2\pi}{3a^2}exp\left(i\frac{E_2-E_3}{\hbar}t\right)\left(-\frac{3\cdot 2^{\frac{5}{2}}a}{5\pi}\right) =$$
$$i\hbar\frac{8\sqrt{2}}{5a}exp\left(i\frac{E_2-E_3}{\hbar}t\right)$$

$$3 \qquad -i\hbar\frac{2\pi}{3a^2}exp\left(i\frac{E_3-E_2}{\hbar}t\right)\int_0^a \sqrt{2}\cdot 2sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{2\pi x}{a}\right)dx =$$
$$-i\hbar\frac{2\pi}{3a^2}exp\left(i\frac{E_3-E_2}{\hbar}t\right)\left(\frac{3\cdot 2^{\frac{5}{2}}a}{5\pi}\right) =$$
$$-i\hbar\frac{8\sqrt{2}}{5a}exp\left(i\frac{E_3-E_2}{\hbar}t\right)$$

$$4 \qquad -i\hbar\frac{2\pi}{3a^2}\int_0^a 6\cdot sin\left(\frac{3\pi x}{a}\right)cos\left(\frac{3\pi x}{a}\right)dx = 0$$

We get the sum of integrals:

$$i\hbar\frac{8\sqrt{2}}{5a}exp\left(i\frac{E_2-E_3}{\hbar}t\right)-i\hbar\frac{8\sqrt{2}}{5a}exp\left(i\frac{E_3-E_2}{\hbar}t\right)=$$

$$-\hbar\frac{16\sqrt{2}}{5a}\cdot sin\left(\frac{E_2-E_3}{\hbar}t\right)$$

We replace the energy difference E_2-E_3:

$$E_2-E_3=\frac{4\hbar^2\pi^2}{ma^2}-\frac{9\hbar^2\pi^2}{2ma^2}=-\frac{5\hbar^2\pi^2}{2ma^2}$$

We get:

$$\langle p\rangle=-\hbar\frac{16\sqrt{2}}{5a}\cdot sin\left(-\frac{5\hbar\pi^2}{2ma^2}t\right)$$

With this we can check the Ehrenfest theorem:

$$\frac{d}{dt}\langle x\rangle=\frac{1}{m}\langle p\rangle$$

$$\langle x\rangle=a\left(\frac{1}{2}-\frac{32\sqrt{2}}{25\pi^2}cos\left(-\frac{5\hbar\pi^2}{2ma^2}t\right)\right)$$

$$\frac{d}{dt}\langle x\rangle=\frac{d}{dt}\left(-\frac{32a\sqrt{2}}{25\pi^2}cos\left(-\frac{5\hbar\pi^2}{2ma^2}t\right)\right)=\frac{32a\sqrt{2}}{25\pi^2}sin\left(-\frac{5\hbar\pi^2}{2ma^2}t\right)\left(-\frac{5\hbar\pi^2}{2ma^2}\right)=$$

$$-\frac{16\hbar\sqrt{2}}{5ma}sin\left(-\frac{5\hbar\pi^2}{2ma^2}t\right)=\frac{1}{m}\langle p\rangle$$