In quantum mechanics observables are real. As we work with complex numbers, results are often presented in terms of sin and cos. During the process of calculation these are not visible.

This paper takes trigonometric identities and translate them to the according exponentials in order to help detecting them during messy calculations. You may also find that there is a lot more of trigonometric identities you can produce with exponentials.

Hope I can help you with learning quantum mechanics.

Contents

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i}$$

$$\sin(-\theta) = \frac{e^{i(-\theta)} - e^{-i(-\theta)}}{2 \cdot i} = -\frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i} = -\sin(\theta)$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cos(-\theta) = \frac{e^{i(-\theta)} + e^{-i(-\theta)}}{2} = \frac{e^{-i\theta} + e^{i\theta}}{2} = \cos(\theta)$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i}}{\frac{e^{i\theta} + e^{-i\theta}}{2}} = -i \cdot \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{\frac{e^{i(-\theta)} - e^{-i(-\theta)}}{2 \cdot i}}{\frac{e^{i(-\theta)} + e^{-i(-\theta)}}{2}} = i \cdot \frac{e^{-i\theta} - e^{i\theta}}{e^{-i\theta} + e^{i\theta}} = -\tan(\theta)$$

$$\left(\sin(\theta)\right)^2 = \frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i} \cdot \frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i} = -\frac{1}{4}\left(e^{i2\theta} - 2e^{i\theta}e^{-i\theta} + e^{-i2\theta}\right) = -\frac{1}{4}\left(e^{i2\theta} - 2 + e^{-i2\theta}\right) = \frac{1}{2} - \frac{1}{4}\left(e^{i2\theta} + e^{-i2\theta}\right)$$

$$\left(\cos(\theta)\right)^2 = \frac{e^{i\theta} + e^{-i\theta}}{2} \cdot \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{4}\left(e^{i2\theta} + 2e^{i\theta}e^{-i\theta} + e^{-i2\theta}\right) = \frac{1}{4}\left(e^{i2\theta} + 2 + e^{-i2\theta}\right) = \frac{1}{2} + \frac{1}{4}\left(e^{i2\theta} + e^{-i2\theta}\right)$$

Note: This example shows the symmetry of $\left(\sin(\theta)\right)^2$ and $\left(\cos(\theta)\right)^2$ to $\frac{1}{2}$.

$$\left(\sin(\theta)\right)^2 + \left(\cos(\theta)\right)^2 = \frac{1}{2} - \frac{1}{4}\left(e^{i2\theta} + e^{-i2\theta}\right) + \frac{1}{2} + \frac{1}{4}\left(e^{i2\theta} + e^{-i2\theta}\right) = 1$$

Alternativ:

$$\left(\sin(\theta)\right)^2 + \left(\cos(\theta)\right)^2 = \frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i}\frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i} + \frac{e^{i\theta} + e^{-i\theta}}{2}\frac{e^{i\theta} + e^{-i\theta}}{2} =$$

$$\frac{1}{4}\left(\left(e^{i2\theta} + 2e^{i\theta}e^{-i\theta} + e^{-i2\theta}\right) - \left(e^{i2\theta} - 2e^{i\theta}e^{-i\theta} + e^{-i2\theta}\right)\right) =$$

$$\frac{4}{4}\left(e^{i\theta}e^{-i\theta}\right) = 1$$

$$sin(2\cdot\theta)=\frac{e^{i2\theta}-e^{-i2\theta}}{2\cdot i}=\frac{(e^{i\theta})^2-(e^{-i\theta})^2}{2\cdot i}=\frac{(e^{i\theta}-e^{-i\theta})(e^{i\theta}+e^{-i\theta})}{2\cdot i}=$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\cdot\left(e^{i\theta}+e^{-i\theta}\right)=2\cdot\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}=$$

$$2\cdot sin(\theta)\cdot cos(\theta)$$

$$cos(2\cdot\theta)=\frac{e^{i2\theta}+e^{-i2\theta}}{2}=\frac{(e^{i\theta})^2+(e^{-i\theta})^2}{2}=\frac{(e^{i\theta})^2+2+(e^{-i\theta})^2-2}{2}=$$

$$\frac{\left(e^{i\theta}\right)^2+2\cdot e^{i\theta}e^{-i\theta}+\left(e^{-i\theta}\right)^2-2}{2}=\frac{\left(e^{i\theta}+e^{-i\theta}\right)^2}{2}-1=$$

$$2\cdot\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2-1=$$

$$2\cdot\left(cos(\theta)\right)^2-1$$

Alternative:

$$cos(2\cdot\theta)=\frac{e^{i2\theta}+e^{-i2\theta}}{2}=\frac{\left(e^{i\theta}\right)^2+\left(e^{-i\theta}\right)^2}{2}=$$

$$\frac{\left(e^{i\theta}\right)^2+\left(e^{-i\theta}\right)^2+\left(e^{i\theta}\right)^2+\left(e^{-i\theta}\right)^2}{2\cdot 2}=$$

$$\frac{\left(e^{i\theta}\right)^2+2\cdot e^{i\theta}e^{-i\theta}+\left(e^{-i\theta}\right)^2+\left(e^{i\theta}\right)^2-2e^{i\theta}e^{-i\theta}+\left(e^{-i\theta}\right)^2}{2\cdot 2}=$$

$$\frac{\left(e^{i\theta}\right)^2+2e^{i\theta}e^{-i\theta}+\left(e^{-i\theta}\right)^2}{2\cdot 2}+\frac{\left(e^{i\theta}\right)^2-2e^{i\theta}e^{-i\theta}+\left(e^{-i\theta}\right)^2}{2\cdot 2}=$$

$$\frac{\left(e^{i\theta}\right)^2+2e^{i\theta}e^{-i\theta}+\left(e^{-i\theta}\right)^2}{2\cdot 2}-\frac{\left(e^{i\theta}\right)^2-2e^{i\theta}e^{-i\theta}+\left(e^{-i\theta}\right)^2}{-(2\cdot 2)}=$$

$$\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2-\left(\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\right)^2=$$

$$\left(cos(\theta)\right)^2-\left(sin(\theta)\right)^2$$

$$tan(2\cdot\theta)=\frac{sin(2\cdot\theta)}{cos(2\cdot\theta)}=\frac{2\cdot\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}}{\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2-\left(\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\right)^2}=$$

$$\frac{2\cdot\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}}{\frac{e^{i2\theta}+2e^{i\theta}e^{-i\theta}+e^{-i2\theta}}{4}+\frac{e^{i2\theta}-2e^{i\theta}e^{-i\theta}+e^{-i2\theta}}{4}}=$$

$$\frac{8\cdot\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}}{e^{i2\theta}+2e^{i\theta}e^{-i\theta}+e^{-i2\theta}+e^{i2\theta}-2e^{i\theta}e^{-i\theta}+e^{-i2\theta}}=$$

$$\frac{2\cdot -i\cdot\left(e^{i\theta}-e^{-i\theta}\right)\cdot\left(e^{i\theta}+e^{-i\theta}\right)}{e^{i2\theta}+e^{-i2\theta}+e^{i2\theta}+e^{-i2\theta}}=$$

$$\frac{2\cdot\left(\frac{-i\cdot\left(e^{i\theta}-e^{-i\theta}\right)\cdot\left(e^{i\theta}+e^{-i\theta}\right)}{(e^{i\theta}+e^{-i\theta})^2}\right)}{\left(\frac{e^{i2\theta}+e^{-i2\theta}+e^{i2\theta}+e^{-i2\theta}}{(e^{i\theta}+e^{-i\theta})^2}\right)}=$$

$$\frac{2\cdot\left(\frac{-i\cdot\left(e^{i\theta}-e^{-i\theta}\right)}{e^{i\theta}+e^{-i\theta}}\right)}{\left(\frac{e^{i2\theta}+e^{-i2\theta}+e^{i2\theta}+e^{-i2\theta}}{(e^{i\theta}+e^{-i\theta})^2}\right)}=$$

$$\frac{2\cdot tan(\theta)}{\frac{2(e^{i2\theta}+e^{-i2\theta})}{(e^{i\theta}+e^{-i\theta})^2}}=$$

$$\frac{2\cdot tan(\theta)}{\frac{2(e^{i2\theta}-2e^{i\theta}e^{-i\theta}+e^{-i2\theta})+4e^{i\theta}e^{-i\theta}}{(e^{i\theta}+e^{-i\theta})^2}}=$$

$$\frac{2\cdot tan(\theta)}{\frac{2(e^{i\theta}-e^{-i\theta})^2+4e^{i\theta}e^{-i\theta}}{(e^{i\theta}+e^{-i\theta})^2}}=$$

$$\frac{2\cdot tan(\theta)}{\frac{(e^{i\theta}-e^{-i\theta})^2}{(e^{i\theta}+e^{-i\theta})^2}+\frac{(e^{i\theta}-e^{-i\theta})^2}{(e^{i\theta}+e^{-i\theta})^2}+\frac{4e^{i\theta}e^{-i\theta}}{(e^{i\theta}+e^{-i\theta})^2}}=$$

$$\frac{2\cdot tan(\theta)}{-\left(tan(\theta)\right)^2+\frac{(e^{i\theta}-e^{-i\theta})^2+4e^{i\theta}e^{-i\theta}}{(e^{i\theta}+e^{-i\theta})^2}}=$$

$$\frac{2\cdot tan(\theta)}{-\left(tan(\theta)\right)^2+\frac{e^{i2\theta}-2e^{i\theta}e^{-i\theta}+e^{-i2\theta}+4e^{i\theta}e^{-i\theta}}{(e^{i\theta}+e^{-i\theta})^2}}=$$

$$\frac{2\cdot tan(\theta)}{-\left(tan(\theta)\right)^2+\frac{(e^{i\theta}+e^{-i\theta})^2}{(e^{i\theta}+e^{-i\theta})^2}}=$$

$$\frac{2\cdot tan(\theta)}{1-\left(tan(\theta)\right)^2}$$

$$sin(\theta)+sin(\varphi)=\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}+\frac{e^{i\varphi}-e^{-i\varphi}}{2\cdot i}=$$

$$\frac{e^{i\theta}+e^{i\varphi}-e^{-i\theta}-e^{-i\varphi}}{2\cdot i}=$$

$$2\cdot\frac{e^{i\theta}+e^{i\varphi}-e^{-i\theta}-e^{-i\varphi}}{4\cdot i}=$$

$$2\cdot\frac{e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}+e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}}{4\cdot i}=$$

$$2\cdot\frac{e^{i\left(\frac{\theta+\varphi}{2}\right)}\left(e^{i\left(\frac{\theta-\varphi}{2}\right)}+e^{-i\left(\frac{\theta-\varphi}{2}\right)}\right)-e^{-i\left(\frac{\theta+\varphi}{2}\right)}\left(e^{-i\left(\frac{\theta-\varphi}{2}\right)}+e^{i\left(\frac{\theta-\varphi}{2}\right)}\right)}{4\cdot i}=$$

$$2\cdot\frac{e^{i\left(\frac{\theta+\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}}{2\cdot i}\cdot\frac{e^{i\left(\frac{\theta-\varphi}{2}\right)}+e^{-i\left(\frac{\theta-\varphi}{2}\right)}}{2}=$$

$$2\cdot sin\left(\frac{\theta+\varphi}{2}\right)\cdot cos\left(\frac{\theta-\varphi}{2}\right)$$

$$sin(\theta)-sin(\varphi)=\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}-\frac{e^{i\varphi}-e^{-i\varphi}}{2\cdot i}=$$

$$\frac{e^{i\theta}-e^{-i\theta}-e^{i\varphi}+e^{-i\varphi}}{2\cdot i}=$$

$$\frac{e^{i\theta}+e^{-i\varphi}-e^{-i\theta}-e^{i\varphi}}{2\cdot i}=$$

$$2\cdot\frac{e^{i\theta}+e^{-i\varphi}-e^{-i\theta}-e^{i\varphi}}{4\cdot i}=$$

$$2\cdot\frac{e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}-e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}}{4\cdot i}=$$

$$2\cdot\frac{e^{i\left(\frac{\theta-\varphi}{2}\right)}\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)-e^{-i\left(\frac{\theta-\varphi}{2}\right)}\left(e^{-i\left(\frac{\theta+\varphi}{2}\right)}+e^{i\left(\frac{\theta+\varphi}{2}\right)}\right)}{4\cdot i}=$$

$$2\cdot\frac{e^{i\left(\frac{\theta-\varphi}{2}\right)}-e^{-i\left(\frac{\theta-\varphi}{2}\right)}}{2\cdot i}\cdot\frac{e^{-i\left(\frac{\theta+\varphi}{2}\right)}+e^{i\left(\frac{\theta+\varphi}{2}\right)}}{2}=$$

$$2\cdot sin\left(\frac{\theta-\varphi}{2}\right)\cdot cos\left(\frac{\theta+\varphi}{2}\right)$$

$$cos(\theta)+cos(\varphi)=\frac{e^{i\theta}+e^{-i\theta}}{2}+\frac{e^{i\varphi}+e^{-i\varphi}}{2}=$$

$$\frac{e^{i\theta}+e^{-i\theta}+e^{i\varphi}+e^{-i\varphi}}{2}=$$

$$\frac{e^{i\theta}+e^{i\varphi}+e^{-i\theta}+e^{-i\varphi}}{2}=$$

$$\frac{e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}+e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}}{2}=$$

$$\frac{e^{i\left(\frac{\theta+\varphi}{2}\right)}\left(e^{i\left(\frac{\theta-\varphi}{2}\right)}+e^{-i\left(\frac{\theta-\varphi}{2}\right)}\right)+e^{-i\left(\frac{\theta+\varphi}{2}\right)}\left(e^{-i\left(\frac{\theta-\varphi}{2}\right)}+e^{i\left(\frac{\theta-\varphi}{2}\right)}\right)}{2}=$$

$$\frac{\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)\left(e^{-i\left(\frac{\theta-\varphi}{2}\right)}+e^{i\left(\frac{\theta-\varphi}{2}\right)}\right)}{2}=$$

$$2\cdot\frac{\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)\left(e^{-i\left(\frac{\theta-\varphi}{2}\right)}+e^{i\left(\frac{\theta-\varphi}{2}\right)}\right)}{2\cdot 2}=$$

$$2\cdot\frac{\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)}{2}\cdot\frac{\left(e^{-i\left(\frac{\theta-\varphi}{2}\right)}+e^{i\left(\frac{\theta-\varphi}{2}\right)}\right)}{2}=$$

$$cos\left(\frac{\theta+\varphi}{2}\right)\cdot cos\left(\frac{\theta-\varphi}{2}\right)$$

$$cos(\theta)-cos(\varphi)=\frac{e^{i\theta}+e^{-i\theta}}{2}-\frac{e^{i\varphi}+e^{-i\varphi}}{2}=$$

$$\frac{e^{i\theta}+e^{-i\theta}-e^{i\varphi}-e^{-i\varphi}}{2}=$$

$$\frac{e^{i\theta}-e^{-i\varphi}+e^{-i\theta}-e^{i\varphi}}{2}=$$

$$\frac{e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{i\left(\frac{\theta-\varphi}{2}\right)}+e^{-i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}-e^{i\left(\frac{\theta+\varphi}{2}\right)}e^{-i\left(\frac{\theta-\varphi}{2}\right)}}{2}=$$

$$\frac{e^{i\left(\frac{\theta-\varphi}{2}\right)}\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)-e^{-i\left(\frac{\theta-\varphi}{2}\right)}\left(-e^{-i\left(\frac{\theta+\varphi}{2}\right)}+e^{i\left(\frac{\theta+\varphi}{2}\right)}\right)}{2}=$$

$$\frac{\left(e^{i\left(\frac{\theta-\varphi}{2}\right)}-e^{-i\left(\frac{\theta-\varphi}{2}\right)}\right)\left(-e^{-i\left(\frac{\theta+\varphi}{2}\right)}+e^{i\left(\frac{\theta+\varphi}{2}\right)}\right)}{2}=$$

$$\frac{\left(e^{i\left(\frac{\theta-\varphi}{2}\right)}-e^{-i\left(\frac{\theta-\varphi}{2}\right)}\right)\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)}{2}=$$

$$2\cdot i\cdot i\cdot\frac{\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)\left(e^{-i\left(\frac{\theta-\varphi}{2}\right)}-e^{i\left(\frac{\theta-\varphi}{2}\right)}\right)}{2\cdot i\cdot 2\cdot i}=$$

$$-2\cdot\frac{\left(e^{i\left(\frac{\theta+\varphi}{2}\right)}-e^{-i\left(\frac{\theta+\varphi}{2}\right)}\right)}{2\cdot i}\cdot\frac{\left(e^{-i\left(\frac{\theta-\varphi}{2}\right)}-e^{i\left(\frac{\theta-\varphi}{2}\right)}\right)}{2\cdot i}=$$

$$-2\cdot sin\left(\frac{\theta+\varphi}{2}\right)\cdot sin\left(\frac{\theta-\varphi}{2}\right)$$

$$tan(\theta)+tan(\varphi)=\frac{sin(\theta)}{cos(\theta)}+\frac{sin(\varphi)}{cos(\varphi)}=-i\cdot\left(\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}+\frac{e^{i\varphi}-e^{-i\varphi}}{e^{i\varphi}+e^{-i\varphi}}\right)=$$

$$-i\cdot\frac{\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\varphi}+e^{-i\varphi}\right)+\left(e^{i\varphi}-e^{-i\varphi}\right)\left(e^{i\theta}+e^{-i\theta}\right)}{\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\varphi}+e^{-i\varphi}\right)}=$$

$$-i\cdot\frac{e^{i(\theta+\varphi)}+e^{i(\theta-\varphi)}-e^{i(\varphi-\theta)}-e^{-i(\varphi+\theta)}+e^{i(\theta+\varphi)}+e^{i(\varphi-\theta)}-e^{i(\theta-\varphi)}-e^{-i(\varphi+\theta)}}{\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\varphi}+e^{-i\varphi}\right)}=$$

$$-i \cdot \frac{2 \cdot e^{i(\theta+\varphi)} - 2 \cdot e^{-i(\varphi+\theta)}}{(e^{i\theta} + e^{-i\theta})(e^{i\varphi} + e^{-i\varphi})} =$$

$$-2 \cdot i \cdot \left(e^{i(\theta+\varphi)} - e^{-i(\varphi+\theta)}\right) \cdot \frac{1}{(e^{i\theta} + e^{-i\theta})(e^{i\varphi} + e^{-i\varphi})} =$$

$$\frac{\left(e^{i(\theta+\varphi)} - e^{-i(\varphi+\theta)}\right)}{2 \cdot i} \cdot \frac{2 \cdot 2}{(e^{i\theta} + e^{-i\theta})(e^{i\varphi} + e^{-i\varphi})} =$$

$$sin(\theta + \varphi) \cdot \frac{1}{cos(\theta)cos(\varphi)} =$$

$$\frac{sin(\theta + \varphi)}{cos(\theta)cos(\varphi)}$$

Similar:

$$tan(\theta) - tan(\varphi) = \frac{sin(\theta - \varphi)}{cos(\theta)cos(\varphi)}$$

$$sin(\theta) \cdot sin(\varphi) = \frac{e^{i\theta} - e^{-i\theta}}{2 \cdot i} \cdot \frac{e^{i\varphi} - e^{-i\varphi}}{2 \cdot i} =$$

$$-\frac{e^{i(\theta+\varphi)} - e^{i(\theta-\varphi)} - e^{i(\varphi-\theta)} + e^{-i(\theta+\varphi)}}{4} =$$

$$-\frac{e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)} - e^{i(\theta-\varphi)} - e^{-i(\varphi-\theta)}}{4} =$$

$$-\frac{e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)}}{4} + \frac{e^{i(\theta-\varphi)} + e^{-i(\varphi-\theta)}}{4} =$$

$$\frac{1}{2} \cdot \left(-\frac{e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)}}{2} + \frac{e^{i(\theta-\varphi)} + e^{-i(\varphi-\theta)}}{2}\right) =$$

$$\frac{1}{2} \cdot \left(cos(\theta - \varphi) - cos(\theta + \varphi)\right)$$

$$cos(\theta) \cdot cos(\varphi) = \frac{e^{i\theta} + e^{-i\theta}}{2} \cdot \frac{e^{i\varphi} + e^{-i\varphi}}{2} =$$

$$\frac{e^{i(\theta+\varphi)} + e^{i(\theta-\varphi)} + e^{i(\varphi-\theta)} + e^{-i(\theta+\varphi)}}{4} =$$

$$\frac{e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)} + e^{i(\theta-\varphi)} + e^{-i(\varphi-\theta)}}{4} =$$

$$\frac{e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)}}{4} + \frac{e^{i(\theta-\varphi)} + e^{-i(\varphi-\theta)}}{4} =$$

$$\frac{1}{2} \cdot \left(\frac{e^{i(\theta+\varphi)} + e^{-i(\theta+\varphi)}}{2} + \frac{e^{i(\theta-\varphi)} + e^{-i(\varphi-\theta)}}{2}\right) =$$

$$\frac{1}{2} \cdot \left(cos(\theta + \varphi) + cos(\theta - \varphi)\right)$$

$$sin(\theta)\cdot cos(\varphi)=\frac{e^{i\theta}-e^{-i\theta}}{2\cdot i}\cdot\frac{e^{i\varphi}+e^{-i\varphi}}{2}=$$

$$\frac{e^{i(\theta+\varphi)}+e^{i(\theta-\varphi)}-e^{i(\varphi-\theta)}-e^{-i(\theta+\varphi)}}{4\cdot i}=$$

$$\frac{e^{i(\theta+\varphi)}-e^{-i(\theta+\varphi)}+e^{i(\theta-\varphi)}-e^{-i(\varphi-\theta)}}{4\cdot i}=$$

$$\frac{e^{i(\theta+\varphi)}+e^{-i(\theta+\varphi)}}{4\cdot i}+\frac{e^{i(\theta-\varphi)}+e^{-i(\varphi-\theta)}}{4\cdot i}=$$

$$\frac{1}{2}\cdot\left(\frac{e^{i(\theta+\varphi)}+e^{-i(\theta+\varphi)}}{2\cdot i}+\frac{e^{i(\theta-\varphi)}+e^{-i(\varphi-\theta)}}{2\cdot i}\right)=$$

$$\frac{1}{2}\cdot\big(sin(\theta+\varphi)+sin(\theta-\varphi)\big)$$

$$tan(\theta)\cdot tan(\varphi)=\left(-i\cdot\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right)\cdot\left(-i\cdot\frac{e^{i\varphi}-e^{-i\varphi}}{e^{i\varphi}+e^{-i\varphi}}\right)=$$

$$-\frac{\left(e^{i\theta}-e^{-i\theta}\right)\left(e^{i\varphi}-e^{-i\varphi}\right)}{\left(e^{i\theta}+e^{-i\theta}\right)\left(e^{i\varphi}+e^{-i\varphi}\right)}=$$

$$-\frac{e^{i(\theta+\varphi)}-e^{i(\theta-\varphi)}-e^{i(\varphi-\theta)}+e^{-i(\varphi+\theta)}}{e^{i(\theta+\varphi)}+e^{i(\theta-\varphi)}+e^{i(\varphi-\theta)}+e^{-i(\varphi+\theta)}}=$$

$$-\frac{e^{i(\theta+\varphi)}+e^{-i(\theta+\varphi)}-e^{i(\theta-\varphi)}-e^{-i(\theta-\varphi)}}{e^{i(\theta+\varphi)}+e^{-i(\varphi+\theta)}+e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}}=$$

$$\frac{-e^{i(\theta+\varphi)}-e^{-i(\theta+\varphi)}+e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}}{e^{i(\theta+\varphi)}+e^{-i(\varphi+\theta)}+e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}}=$$

$$-\frac{e^{i(\theta+\varphi)}+e^{-i(\theta+\varphi)}}{e^{i(\theta+\varphi)}+e^{-i(\varphi+\theta)}+e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}}+\frac{e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}}{e^{i(\theta+\varphi)}+e^{-i(\varphi+\theta)}+e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}}=$$

$$-\frac{1}{1+e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}}+\frac{1}{e^{i(\theta+\varphi)}+e^{-i(\varphi+\theta)}+1}=$$

$$-\frac{1}{1+2\cdot cos(\theta-\varphi)}+\frac{1}{2\cdot cos(\theta+\varphi)+1}$$