This paper shows one way to get the 1D Schrödinger wavefunction out of the de Broglie relation.

Hope I can help you with learning quantum mechanics.

Contents

Plane wave

For a plane wave in space holds the differential equation:

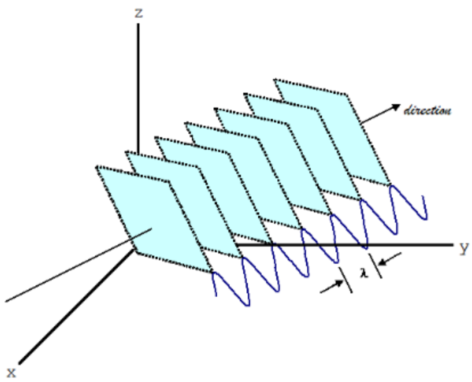
$$a^2 \Delta\, \psi = \frac{\partial^2 \psi}{\partial t^2} \qquad (1)$$

Note: Δ is the Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Note: $\psi(r,t)$ is a function of space and time.

Note: Plane waves are an approximation based on radial waves if the distance to the origin of the wave is large compared to the wave length.



Note: When dealing with plane waves we always can orient the measuring system in a way that the wave propagates along one axis. The Laplace operator reduces to the partial derivatives along this axis.

Note: The vector pointing in the direction of the wave is called wave vector $\vec{k}$ with the magnitude:

$$|\vec{k}| = k = \frac{2\pi}{T} \cdot \frac{1}{c} = 2 \cdot \pi \cdot \nu \cdot \frac{1}{c} = \frac{2 \cdot \pi \cdot \nu}{\nu \cdot \lambda} = \frac{2 \cdot \pi}{\lambda}$$

We orient our system in a way that the wave propagates along the x-axis.

The differential equation (1) then becomes:

$$a^2 \frac{\partial^2}{\partial x^2}\, \psi(x,t) = \frac{\partial^2}{\partial t^2} \psi(x,t) \qquad (2)$$

We try the wave function:

$$\psi(x,t) = A \cdot e^{-i\left(2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right)}$$

Note: A represents the amplitude, $-i\left(2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right)$ the oscillation.

Note: If we fix the position x, the oscillation is caused by changing in time.

Note: If we fix time t, the oscillation is caused by changing the position.

Note: Changing position can be done arbitrarily.

We check whether $\psi(x,t)$ is a solution to the differential equation (2):

$$a^2 \frac{\partial^2}{\partial x^2}\, \psi(x,t) = \frac{\partial^2}{\partial t^2} \psi(x,t)$$

Right side:

$$\frac{\partial^2}{\partial t^2} \psi(x,t) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} \psi(x,t)\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(A \cdot e^{-i\left(2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right)}\right)\right) =$$

$$\frac{\partial}{\partial t}\left(-\frac{i2\pi}{T}\cdot A\cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}\right)=-\frac{4\pi^2}{T^2}\cdot A\cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}=$$

$$-\frac{4\pi^2}{T^2}\psi(x,t)$$

Left side:

$$a^2\frac{\partial^2}{\partial x^2}\,\psi(x,t)=a^2\frac{\partial}{\partial x}\,\frac{\partial}{\partial x}\psi(x,t)=$$

$$a^2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(A\cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}\right)\right)=a^2\frac{\partial}{\partial x}\left(-\frac{i2\pi}{\lambda}\cdot A\cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}\right)=$$

$$a^2\left(-\frac{4\pi^2}{\lambda^2}\cdot A\cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}\right)=-\frac{4\pi^2a^2}{\lambda^2}\left(A\cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}\right)=$$

$$-\frac{4\pi^2a^2}{\lambda^2}\,\psi(x,t)$$

We go back to the differential equation (2):

$$a^2\frac{\partial^2}{\partial x^2}\,\psi=\frac{\partial^2}{\partial t^2}\psi$$

We insert our results:

$$-\frac{4\pi^2a^2}{\lambda^2}\,\psi(x,t)=-\frac{4\pi^2}{T^2}\psi(x,t)$$

We get:

$$\frac{a^24\pi^2}{\lambda^2}=\frac{4\pi^2}{T^2}\;\rightarrow\;a^2=\frac{\lambda^2}{T^2}\;\rightarrow\;a=\frac{\lambda}{T}=\lambda\nu$$

Note: a is the phase velocity of the plane wave. For an electromagnetic wave we have $c=\lambda\nu$ with c being the speed of light.

Note: The group velocity is the velocity of an envelope of the wave – e. g. a wave packet. For more information please see https://en.wikipedia.org/wiki/Group_velocity

Result: $\psi(x,t)$ is a solution to the differential equation (2):

$$\psi(x,t)=A\cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}$$

De Broglie

De Broglie:	Classic physics:
A particle with momentum along the x-axis $$p$$ can be described as a plane wave with wave number $$k = \frac{p}{h} \;\rightarrow\; p = h \cdot k$$ and frequency $$\nu = \frac{E}{h} \;\rightarrow\; E = h \cdot \nu$$	From classic physics we use: $$E = \frac{p^2}{2m}$$

Note: h is the Planck constant.

Note: Working in one dimension we omit vector notation: $\vec{p} = p$ etc.

We combine de Broglie and classic:

$$h \cdot \nu = E = \frac{p^2}{2m} = \frac{h^2 k^2}{2m} \rightarrow \nu = \frac{hk^2}{2m}$$

Note: This is the de Broglie relation.

Note: ν is the frequency of the wave, $\nu = \frac{\omega}{2\pi}$.

Nonrelativistic Schrödinger

We rewrite the classic energy equation:

$$E = \frac{p^2}{2m} \;\rightarrow\; E - \frac{p^2}{2m} = 0$$

We rewrite by use of the de Broglie relation:

$$h \cdot \nu - \frac{h^2 k^2}{2m} = 0$$

$$\nu - \frac{hk^2}{2m} = 0$$

$$\frac{\omega}{2\pi} - \frac{hk^2}{2m} = 0 \qquad (3)$$

We use the wave function:

$$\psi(x,t) = A \cdot e^{-i\left(2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right)}$$

We rewrite the wave function with k and ω:

$$\psi(x,t) = A \cdot e^{-i(\omega \cdot t - k \cdot x)}$$

For this wave function holds:

$\frac{\partial}{\partial x}\psi(x,t) = i \cdot k \cdot \psi(x,t)$	$\frac{\partial}{\partial t}\psi(x,t) = -i \cdot \omega \cdot \psi(x,t)$
Short form: $-i \cdot \frac{\partial}{\partial x} = k$	$i \cdot \frac{\partial}{\partial t} = \omega$
$\frac{\partial^2}{\partial x^2}\psi(x,t) = -k^2 \cdot \psi(x,t)$	$\frac{\partial^2}{\partial t^2}\psi(x,t) = -\omega^2 \cdot \psi(x,t)$
Short form: $-\frac{\partial^2}{\partial x^2} = k^2$	$-\frac{\partial^2}{\partial t^2} = \omega^2$

We take the de Broglie relation (3):

$$\frac{\omega}{2\pi} - \frac{hk^2}{2m} = 0$$

We insert ω and k^2:

$$\omega = i\frac{\partial}{\partial t}, k^2 = -\frac{\partial^2}{\partial x^2}$$

We get:

$$\frac{i}{2\pi}\frac{\partial}{\partial t} + \frac{h}{2m}\frac{\partial^2}{\partial x^2} = 0$$

To get this in the form wanted we modify:

$$\frac{i}{2\pi}\frac{\partial}{\partial t} + \frac{h}{2m}\frac{\partial^2}{\partial x^2} = 0$$

$$i\frac{\partial}{\partial t} + \frac{2\pi h}{2m}\frac{\partial^2}{\partial x^2} = 0$$

By using $\hbar = \frac{h}{2\pi} \rightarrow h = \hbar \cdot 2\pi$, we get:

$$i\frac{\partial}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} = 0$$

This is the nonrelativistic Schrödinger equation in one dimension for a free particle.

Note: Together with:

$$\psi(x,t) = A \cdot e^{-i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}$$

we could also use:

$$\psi(x,t) = A \cdot e^{i\left(2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)\right)}$$

This would represent a particle running into the opposite direction and finally lead to:

$$-i\frac{\partial}{\partial t}+\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}=0$$

From a classic point of view the energy of the particle is either positive or negative, depending on the direction we watch it.