The uncertainty relation for momentum and position in both momentum and position representation.

Related information you may find at:

https://www.doe.carleton.ca/~tjs/475_pdf_02/snew5.pdf

https://quantummechanics.ucsd.edu/ph130a/130_notes/node79.html

Hope I can help you with learning quantum mechanics.

Contents

Position and momentum

The position wave function: $\psi(x) = \langle x\vert\psi\rangle$	The momentum wave function: $\tilde{\psi}(p) = \langle p\vert\psi\rangle$
$\psi(x) = \langle x\vert\psi\rangle = \int_{-\infty}^{\infty} \langle x\vert p\rangle\langle p\vert\psi\rangle dp = \int_{-\infty}^{\infty} \langle p\vert x\rangle\tilde{\psi}(p)dp = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \exp\left(i\frac{p}{\hbar}x\right)\tilde{\psi}(p)dp$ Note: We get $\psi(x)$ by integrating over all possible momenta p.	$\tilde{\psi}(p) = \langle p\vert\psi\rangle = \int_{-\infty}^{\infty} \langle p\vert x\rangle\langle x\vert\psi\rangle dx = \int_{-\infty}^{\infty} \langle p\vert x\rangle\psi(x)dx = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \exp\left(-i\frac{p}{\hbar}x\right)\psi(x)dx$ Note: We get $\tilde{\psi}(p)$ by integrating over all possible coordinates x.

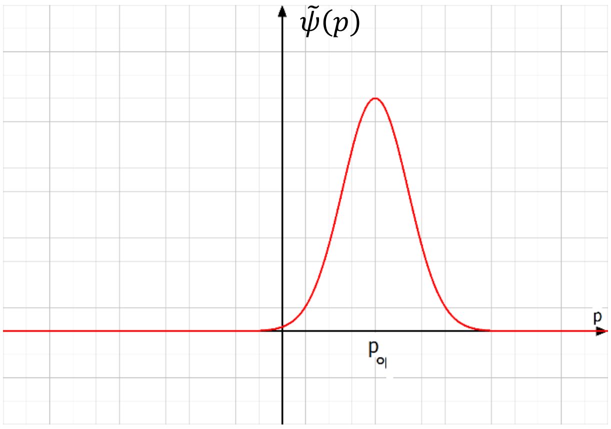
Note: inserting $\vert p\rangle\langle p\vert$ resp. $\vert x\rangle\langle x\vert$ is called resolving the identity.

The effect of the position operator X and momentum operator P in position representation:	The effect of the position operator X and momentum operator P in momentum representation:
$X\psi(x) = x\cdot\psi(x)$ $P\psi(x) = -i\hbar\frac{d}{dx}\psi(x)$	$X\tilde{\psi}(p) = i\hbar\frac{d}{dp}\tilde{\psi}(p)$ $P\tilde{\psi}(p) = p\cdot\tilde{\psi}(p)$

We use a Gauss wave function in momentum representation:

$$\tilde{\psi}(p) = \frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)$$

The shape of the function looks like:



The factor $\frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}}$ chosen for the function to be normalized:

$$\langle\psi\vert\psi\rangle = \int_{-\infty}^{\infty} \left\vert\tilde{\psi}(p)\right\vert^2 dp = 1$$

Calculating the momentum uncertainty by use of the momentum wave function

We calculate the expectation value $\langle P\rangle_\psi$ in momentum representation:

$$\langle P\rangle_\psi = \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} p\cdot\left|\tilde{\psi}(p)\right|^2 dp =$$

$$\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} p\cdot exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)dp =;$$

We replace:

$$p-p_0 \coloneqq q \to p = q+p_0, dp \to dq$$

$$\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} (q+p_0)\cdot exp\left(-\frac{\sigma^2(q)^2}{\hbar^2}\right)dq =$$

$$\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} q\cdot exp\left(-\frac{\sigma^2(q)^2}{\hbar^2}\right)dq + \frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} p_0\cdot exp\left(-\frac{\sigma^2(q)^2}{\hbar^2}\right)dq =$$

$$0 + \frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} p_0\cdot exp\left(-\frac{\sigma^2(q)^2}{\hbar^2}\right)dq =$$

$$p_0\cdot\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(q)^2}{\hbar^2}\right)dq =$$

$$p_0\cdot\frac{\sigma}{\hbar\sqrt{\pi}}\cdot\frac{\hbar\sqrt{\pi}}{\sigma} =$$

$$p_0$$

We get:

$$\langle P\rangle_\psi = p_0$$

We calculate the expectation value $\langle P^2\rangle_\psi$:

$$\langle P^2\rangle_\psi = \langle\psi|P^2|\psi\rangle =$$

$$\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} p^2\cdot exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)dp =;$$

We replace:

$$p-p_0 \coloneqq q \to p = q+p_0, dp \to dq$$

$$\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} (q+p_0)^2\cdot exp\left(-\frac{\sigma^2 q^2}{\hbar^2}\right)dq =$$

$$\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} (q^2+2p_0q+{p_0}^2)\cdot exp\left(-\frac{\sigma^2 q^2}{\hbar^2}\right)dq =$$

$$\frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} q^2\cdot exp\left(-\frac{\sigma^2 q^2}{\hbar^2}\right)dq + \frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} 2p_0q\cdot exp\left(-\frac{\sigma^2 q^2}{\hbar^2}\right)dq$$

$$= + \frac{\sigma}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} {p_0}^2\cdot exp\left(-\frac{\sigma^2 q^2}{\hbar^2}\right)dq =$$

$$\frac{\sigma\hbar^3\sqrt{\pi}}{2\sigma^3\hbar\sqrt{\pi}} + 0 + \frac{{p_0}^2\sigma\hbar\sqrt{\pi}}{\sigma\hbar\sqrt{\pi}} =$$

$$\frac{\hbar^2}{2\sigma^2} + {p_0}^2$$

We get:

$$\langle P^2 \rangle_\psi = \frac{\hbar^2}{2\sigma^2} + {p_0}^2$$

We get the uncertainty for the momentum:

$$(\Delta P)_\psi = \sqrt{\langle P^2 \rangle_\psi - {\langle P \rangle^2}_\psi} = \frac{\hbar}{\sigma\sqrt{2}}$$

Note: this is the statistical variance.

Calculating the position uncertainty by use of the position wave function

We calculate the Gauss wave function in position representation.

We calculate the wave function in position representation:

$$\psi(x) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} exp\left(i\frac{p}{\hbar}x\right)\tilde{\psi}(p)dp =$$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2} + i\frac{p}{\hbar}x\right)dp =;$$

We substitute $q := p - p_0 \rightarrow p = q + p_0; dp \rightarrow dq$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2 q^2}{2\hbar^2} + i\frac{q+p_0}{\hbar}x\right)dq =$$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2 q^2}{2\hbar^2} + i\frac{q}{\hbar}x + i\frac{p_0}{\hbar}x\right)dq =$$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}} exp\left(i\frac{p_0}{\hbar}x\right)\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2 q^2}{2\hbar^2} + i\frac{q}{\hbar}x\right)dq =$$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}} exp\left(i\frac{p_0}{\hbar}x\right)\int_{-\infty}^{\infty} exp - \left(\frac{\sigma^2 q^2}{2\hbar^2} - i\frac{q}{\hbar}x\right)dq =;$$

We work with the inner expression $\left(\frac{\sigma^2 q^2}{2\hbar^2} - i\frac{q}{\hbar}x\right)$:

$$\left(\frac{\sigma^2 q^2}{2\hbar^2} - i\frac{q}{\hbar}x\right) =$$

$$\left(q^2 \cdot \frac{\sigma^2}{2\hbar^2} - q \cdot x \cdot \frac{i}{\hbar}\right) =$$

$$\frac{\sigma^2}{2\hbar^2}\left(q^2 - q \cdot x \cdot \frac{i}{\hbar} \cdot \frac{2\hbar^2}{\sigma^2}\right) =$$

$$\frac{\sigma^2}{2\hbar^2}\left(q^2 - q\cdot x\cdot\frac{i2\hbar}{\sigma^2}\right) =$$

$$\frac{\sigma^2}{2\hbar^2}\left(q^2 - 2\cdot q\cdot x\cdot\frac{i\hbar}{\sigma^2}\right) =$$

$$\frac{\sigma^2}{2\hbar^2}\left(q^2 - 2\cdot q\cdot x\cdot\frac{i\hbar}{\sigma^2} - x^2\cdot\frac{\hbar^2}{\sigma^4} + x^2\cdot\frac{\hbar^2}{\sigma^4}\right) =$$

$$\frac{\sigma^2}{2\hbar^2}\left(q^2 - 2\cdot q\cdot x\cdot\frac{i\hbar}{\sigma^2} - x^2\cdot\frac{\hbar^2}{\sigma^4}\right) + x^2\cdot\frac{\hbar^2}{\sigma^4}\cdot\frac{\sigma^2}{2\hbar^2} =$$

$$\frac{\sigma^2}{2\hbar^2}\left(q^2 - 2\cdot q\cdot x\cdot\frac{i\hbar}{\sigma^2} - x^2\cdot\frac{\hbar^2}{\sigma^4}\right) + \frac{x^2}{2\sigma^2} =$$

$$\frac{\sigma^2}{2\hbar^2}\left(q - x\cdot\frac{i\hbar}{\sigma^2}\right)^2 + \frac{x^2}{2\sigma^2}$$

We insert the result:

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}} exp\left(i\frac{p_0}{\hbar}x\right)\int_{-\infty}^{\infty} exp - \left(\frac{\sigma^2 q^2}{2\hbar^2} - i\frac{q}{\hbar}x\right) dq =$$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}} exp\left(i\frac{p_0}{\hbar}x\right)\int_{-\infty}^{\infty} exp - \left(\frac{\sigma^2}{2\hbar^2}\left(q - x\cdot\frac{i\hbar}{\sigma^2}\right)^2 + \frac{x^2}{2\sigma^2}\right) dq =$$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\int_{-\infty}^{\infty} exp - \left(\frac{\sigma^2}{2\hbar^2}\left(q - x\cdot\frac{i\hbar}{\sigma^2}\right)^2\right) dq =;$$

Note: shifting the Gauss integral along the imaginary axis doesn’t change the result of the integral.

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2 q^2}{2\hbar^2}\right) dq =$$

$$\frac{\sqrt{\sigma}}{\hbar\pi^{\frac{3}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\frac{\hbar\sqrt{2\pi}}{\sigma} =$$

$$\frac{\sqrt{2}}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)$$

Result: We get the position wave function $\psi(x)$ in position representation:

$$\psi(x) = \frac{\sqrt{2}}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)$$

$$\psi^*(x) = \frac{\sqrt{2}}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(-i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)$$

We normalize:

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = 1$$

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx =$$

$$\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(-i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\frac{\sqrt{2}}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)dx =$$

$$\frac{2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2\sigma^2}\right)dx =$$

$$\frac{2}{\sigma\sqrt{\pi}} \cdot \sqrt{\sigma^2\pi} = 2$$

We must correct the factor and get the normalized function:

$$\psi(x) = \frac{1}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)$$

$$\psi^*(x) = \frac{1}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(-i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)$$

This is a Gauss function, oscillating by $exp\left(i\frac{p_0}{\hbar}x\right)$. The oscillation does not change the expectation values $\langle X\rangle$ and $\langle X^2\rangle$.

We calculate the expectation value of the position operator in position representation $\langle X\rangle_\psi$:

$$\langle X\rangle_\psi = \langle\psi|X|\psi\rangle =$$

$$\int_{-\infty}^{\infty} x|\psi(x)|^2 dx = \int_{-\infty}^{\infty} x\psi^*(x)\psi(x)dx =$$

$$\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(-i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)dx =$$

$$\frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} x \cdot exp\left(-i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right) \cdot exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)dx =$$

$$\frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} x \cdot exp\left(-\frac{x^2}{\sigma^2}\right)dx = 0$$

We get:

$$\langle X\rangle_\psi = 0$$

We calculate $\langle X^2\rangle_\psi$:

$$\langle X^2\rangle_\psi = \langle\psi|X^2|\psi\rangle =$$

$$\int_{-\infty}^{\infty} x^2|\psi(x)|^2 dx = \int_{-\infty}^{\infty} x^2\psi^*(x)\psi(x)dx =$$

$$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(-i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{\sigma}\pi^{\frac{1}{4}}} exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)dx =$$

$$\frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 \cdot exp\left(-i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right) \cdot exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)dx =$$

$$\frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 \cdot exp\left(-\frac{x^2}{\sigma^2}\right)dx =$$

$$\frac{1}{\sigma\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{\sigma^6\pi} =$$

$$\frac{1}{\sigma\sqrt{\pi}} \cdot \frac{1}{2}\sigma^3\sqrt{\pi} = \frac{\sigma^2}{2}$$

We get:

$$\langle X^2\rangle_\psi = \frac{\sigma^2}{2}$$

The uncertainty:

$$(\Delta X)_\psi = \sqrt{\langle X^2\rangle_\psi - \langle X\rangle^2{}_\psi} = \frac{\sigma}{\sqrt{2}}$$

We get the uncertainty relation:

$$(\Delta P)_\psi(\Delta X)_\psi =$$

$$\frac{\sigma}{\sqrt{2}} \cdot \frac{\hbar}{\sqrt{2}\sigma} = \frac{\hbar}{2}$$

Calculating the position uncertainty by use of the momentum wave function

We calculate the expectation value of position operator $\langle X\rangle_{\tilde{\psi}}$ using the momentum wave function $\tilde{\psi}(p)$.

$$\tilde{\psi}(p) = \frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)$$

Note: $\tilde{\psi}(p) = \tilde{\psi}^*(p)$

$$\langle X\rangle_{\tilde{\psi}} = \langle\tilde{\psi}|X|\tilde{\psi}\rangle =;$$

Note: In the momentum representation the position operator X differentiates and we need to add the factor $i\hbar$:

$$i\hbar\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\frac{d}{dp}\tilde{\psi}(p)dp =$$

$$i\hbar\int_{-\infty}^{\infty} \frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\frac{d}{dp}\left[\frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\right]dp =$$

$$i\hbar\frac{\sigma}{\pi^{\frac{1}{2}}\hbar}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\frac{d}{dp}\left[exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\right]dp =$$

$$i\hbar\frac{\sigma}{\pi^{\frac{1}{2}}\hbar}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\left(\frac{\sigma^2(p-p_0)}{\hbar^2}\right)dp =$$

$$i\hbar\frac{\sigma}{\pi^{\frac{1}{2}}\hbar}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)\left(\frac{\sigma^2(p-p_0)}{\hbar^2}\right)dp =$$

$$i\hbar\frac{\sigma^3}{\pi^{\frac{1}{2}}\hbar^3}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)\big((p-p_0)\big)dp =;$$

Note: Shifting from p to $p - p_0$ doesn't change the value of the integral.

$$\sim i\hbar\frac{\sigma^3}{\pi^{\frac{1}{2}}\hbar^3}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2 p^2}{\hbar^2}\right)(p)dp = 0$$

We get:

$$\langle X\rangle_{\tilde{\psi}} = 0$$

We calculate $\langle X^2\rangle_{\tilde{\psi}}$ using the momentum wave function $\tilde{\psi}(p)$.

$$\tilde{\psi}(p) = \frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)$$

$$\langle X^2\rangle_{\tilde{\psi}} = \langle\tilde{\psi}|X^2|\tilde{\psi}\rangle =;$$

Note: In the momentum representation the position operator X differentiates and we need to add the factor $i\hbar$ squared:

$$(i\hbar)^2\int_{-\infty}^{\infty}\tilde{\psi}^*(p)\frac{d}{dp}\left(\frac{d}{dp}\tilde{\psi}(p)\right)dp =$$

$$-\hbar^2\int_{-\infty}^{\infty}\frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\frac{d}{dp}\left\{\frac{d}{dp}\left[\frac{\sqrt{\sigma}}{\pi^{\frac{1}{4}}\sqrt{\hbar}} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\right]\right\}dp =$$

$$-\frac{\sigma\hbar}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\frac{d}{dp}\left\{\frac{d}{dp}\left[exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\right]\right\}dp =$$

$$-\frac{\sigma\hbar}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\frac{d}{dp}\left\{exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\cdot\left(-\frac{\sigma^2(p-p_0)}{\hbar^2}\right)\right\}dp =$$

$$-\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\frac{d}{dp}\left\{exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\cdot(p_0-p)\right\}dp =$$

$$-\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\left\{exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\cdot\left(-\frac{\sigma^2(p-p_0)}{\hbar^2}\right)\cdot(p_0-p) - exp\left(-\frac{\sigma^2(p-p_0)^2}{2\hbar^2}\right)\right\}dp =$$

$$-\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)\cdot\left(-\frac{\sigma^2(p-p_0)}{\hbar^2}\right)\cdot(p_0-p) - exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)dp =$$

$$-\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)\cdot\left(-\frac{\sigma^2(p-p_0)}{\hbar^2}\right)\cdot(p_0-p)dp$$
$$+\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)dp =$$

$$-\frac{\sigma^5}{\hbar^3\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)\cdot(p-p_0)^2dp+\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2(p-p_0)^2}{\hbar^2}\right)dp =$$

Note: Shifting from $p-p_0$ to p doesn't change the value of the integral.

$$\sim -\frac{\sigma^5}{\hbar^3\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2p^2}{\hbar^2}\right)\cdot p^2dp+\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} exp\left(-\frac{\sigma^2p^2}{\hbar^2}\right)dp =$$

$$-\frac{\sigma^5}{\hbar^3\pi^{\frac{1}{2}}}\cdot\frac{\hbar^3\pi^{\frac{1}{2}}}{2\sigma^3}+\frac{\sigma^3}{\hbar\pi^{\frac{1}{2}}}\cdot\frac{\hbar\pi^{\frac{1}{2}}}{\sigma} =$$

$$-\frac{\sigma^2}{2}+\sigma^2=\frac{\sigma^2}{2}$$

We get:

$$\langle X^2\rangle_{\tilde{\psi}}=\frac{\sigma^2}{2}$$

We calculate the uncertainty:

$$(\Delta X)_{\tilde{\psi}}=\sqrt{\langle X^2\rangle_{\tilde{\psi}}-\langle X\rangle^2_{\tilde{\psi}}}=\frac{\sigma}{\sqrt{2}}$$

Calculating the momentum uncertainty by use of the position wave function

We calculate the expectation value of momentum operator $\langle P\rangle_\psi$ using the position wave function $\psi(x)$.

$$\psi(x)=\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)$$

$$\psi^*(x)=\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)$$

$$\langle P\rangle_\psi=\langle\psi|P|\psi\rangle =;$$

Note: In the position representation the momentum operator P differentiates and we need to add the factor $-i\hbar$:

$$-i\hbar\int_{-\infty}^{\infty}\psi^*(x)\frac{d}{dx}\psi(x)dx =$$

$$-i\hbar\int_{-\infty}^{\infty}\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)\frac{d}{dx}\left(\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)\right)dx =$$

$$-\frac{i\hbar}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)\frac{d}{dx}\left(exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)\right)dx =$$

$$-\frac{i\hbar}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)\left(exp\left(-\frac{x^2}{2\sigma^2}\right)\left(-\frac{x}{\sigma}\right)exp\left(i\frac{p_0}{\hbar}x\right)\right.$$
$$\left.+\,exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)\left(i\frac{p_0}{\hbar}\right)\right)dx =$$

$$-\frac{i\hbar}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)exp\left(-\frac{x^2}{2\sigma^2}\right)\left(-\frac{x}{\sigma}\right)exp\left(i\frac{p_0}{\hbar}x\right)$$
$$+\,exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)\left(i\frac{p_0}{\hbar}\right)dx =$$

$$-\frac{i\hbar}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-\frac{x^2}{2\sigma^2}\right)\left(-\frac{x}{\sigma}\right)+exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-\frac{x^2}{2\sigma^2}\right)\left(i\frac{p_0}{\hbar}\right)dx =$$

$$-\frac{i\hbar}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(-\frac{x}{\sigma}\right)+exp\left(-\frac{x^2}{\sigma^2}\right)\left(i\frac{p_0}{\hbar}\right)dx =$$

$$-\frac{i\hbar}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(-\frac{x}{\sigma}\right)dx-\frac{i\hbar}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(i\frac{p_0}{\hbar}\right)dx =$$

$$\frac{i\hbar}{\sigma^2\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\cdot x dx+\frac{p_0}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)dx =$$

$$0+p_0$$

We get:

$$\langle P\rangle_\psi = p_0$$

We calculate $\langle P^2\rangle_\psi$using the position wave function $\psi(x)$.

$$\psi(x)=\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)$$

$$\psi^*(x)=\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)$$

$$\langle P^2\rangle_\psi = \langle\psi|P^2|\psi\rangle =;$$

Note: In the position representation the momentum operator P differentiates and we need to add the factor $-i\hbar$ squared $\coloneqq -\hbar^2$:

$$-\hbar^2\int_{-\infty}^{\infty}\psi^*(x)\frac{d}{dx}\left(\frac{d}{dx}\psi(x)\right)dx =$$

$$-\hbar^2\int_{-\infty}^{\infty}\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{1}{\sqrt{\sigma\pi^{\frac{1}{4}}}}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)\right)\right]dx =$$

$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty}exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(-i\frac{p_0}{\hbar}x\right)\frac{d}{dx}\left[\frac{d}{dx}\left(exp\left(-\frac{x^2}{2\sigma^2}\right)exp\left(i\frac{p_0}{\hbar}x\right)\right)\right]dx =$$

$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty}exp\left(-\frac{x^2}{2\sigma^2}-i\frac{p_0}{\hbar}x\right)\frac{d}{dx}\left[\frac{d}{dx}\left(exp\left(i\frac{p_0}{\hbar}x-\frac{x^2}{2\sigma^2}\right)\right)\right]dx =$$

$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2\sigma^2} - i\frac{p_0}{\hbar}x\right)\frac{d}{dx}\left(exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(i\frac{p_0}{\hbar} - \frac{x}{\sigma^2}\right)\right)dx =;$$

We derivate:

$$\frac{d}{dx}\left(exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(i\frac{p_0}{\hbar} - \frac{x}{\sigma^2}\right)\right) =$$

$$exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(i\frac{p_0}{\hbar} - \frac{x}{\sigma^2}\right)^2 - exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\frac{1}{\sigma^2} =$$

$$exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(\left(i\frac{p_0}{\hbar} - \frac{x}{\sigma^2}\right)^2 - \frac{1}{\sigma^2}\right) =$$

$$exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(-\frac{{p_0}^2}{\hbar^2} - i\frac{2xp_0}{\hbar\sigma^2} + \frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}\right) =$$

$$exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(-\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right) - i\frac{2xp_0}{\hbar\sigma^2} + \frac{x^2}{\sigma^4}\right)$$

We combine:

$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2\sigma^2} - i\frac{p_0}{\hbar}x\right)\frac{d}{dx}\left(exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(i\frac{p_0}{\hbar} - \frac{x}{\sigma^2}\right)\right)dx =$$

$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{2\sigma^2} - i\frac{p_0}{\hbar}x\right)exp\left(i\frac{p_0}{\hbar}x - \frac{x^2}{2\sigma^2}\right)\left(-\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right) - i\frac{2xp_0}{\hbar\sigma^2} + \frac{x^2}{\sigma^4}\right)dx =$$

$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(-\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right) - i\frac{2xp_0}{\hbar\sigma^2} + \frac{x^2}{\sigma^4}\right)dx =$$

$$\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right)dx + \frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(i\frac{2xp_0}{\hbar\sigma^2}\right)dx$$
$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\frac{x^2}{\sigma^4}dx$$

We deal with the integrals separately.

First integral:

$$\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right)dx =$$

$$\frac{\hbar^2}{\sigma\sqrt{\pi}}\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right)\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)dx =$$

$$\frac{\hbar^2}{\sigma\sqrt{\pi}}\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right)\sigma\sqrt{\pi} = \hbar^2\left(\frac{{p_0}^2}{\hbar^2} + \frac{1}{\sigma^2}\right) =$$

$${p_0}^2 + \frac{\hbar^2}{\sigma^2}$$

Second integral:

$$\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(i\frac{2xp_0}{\hbar\sigma^2}\right)dx = 0$$

Third integral:

$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\frac{x^2}{\sigma^4}dx =$$

$$-\frac{\hbar^2}{\sigma^5\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)x^2 dx =$$

$$-\frac{\hbar^2}{\sigma^5\sqrt{\pi}}\cdot\frac{1}{2}\cdot\sigma^3\sqrt{\pi} =$$

$$-\frac{\hbar^2}{2\sigma^2}$$

We combine:

$$\langle P^2\rangle_\psi = \frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(\frac{{p_0}^2}{\hbar^2}+\frac{1}{\sigma^2}\right)dx + \frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\left(i\frac{2xp_0}{\hbar\sigma^2}\right)dx$$
$$-\frac{\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{x^2}{\sigma^2}\right)\frac{x^2}{\sigma^4}dx =$$

$${p_0}^2+\frac{\hbar^2}{\sigma^2}-\frac{\hbar^2}{2\sigma^2} =$$

$${p_0}^2+\frac{\hbar^2}{2\sigma^2}$$

We get:

$$\langle P^2\rangle_\psi = {p_0}^2+\frac{\hbar^2}{2\sigma^2}$$

The uncertainty of momentum $(\Delta P)_\psi$:

$$(\Delta P)_\psi = \sqrt{\langle P^2\rangle_\psi - {\langle P\rangle^2}_\psi} =$$

$$\sqrt{{p_0}^2+\frac{\hbar^2}{2\sigma^2}-{p_0}^2} = \frac{\hbar}{\sigma\sqrt{2}}$$

The uncertainty relation:

$$(\Delta X)_\psi(\Delta P)_\psi = \frac{\sigma}{\sqrt{2}}\cdot\frac{\hbar}{\sigma\sqrt{2}} = \frac{\hbar}{2}$$