This short paper shows how to calculate uncertainty of position and momentum operator.

Hope I can help you learning quantum mechanics.

The wave function $\psi(x)$ of a free particle in position representation x:

$$\psi(x)$$

Fourier's theorem states that we can reconstruct $\psi(x)$ by a superposition of plane waves $\phi(k)$:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \phi(k) \cdot e^{ikx} dk$$

$\phi(k)$ is the weight of every phase e^{ikx}, the integration of all weighted phases gives back the original function $\psi(x)$.

Fourier's theorem states that we can do the inverse:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \psi(x) \cdot e^{-ikx} dx$$

$\psi(x)$ and $\phi(k)$ are the Fourier transform of each other. Both contain the same information.

A complex valued function is called square integrable if we can integrate it over the whole space:

$$|\psi|^2 = \int_{-\infty}^{\infty} \psi^* \cdot \psi = c < \infty, \qquad |\phi|^2 = \int_{-\infty}^{\infty} \phi^* \cdot \phi = d < \infty$$

We write $|\psi|^2$ and $|\phi|^2$ as:

$$\langle\psi|\psi\rangle\ resp. \langle\phi|\phi\rangle$$

Fourier states that:

$$\langle\psi(x)|\psi(x)\rangle = \langle\phi(k)|\phi(k)\rangle$$

Note: shifting a function from the right position to the left position within a bra-ket pair always uses complex conjugation.

We can normalize the integral to one:

$$\frac{1}{c}\int_{-\infty}^{\infty} \psi^* \cdot \psi = 1, \qquad \frac{1}{d}\int_{-\infty}^{\infty} \phi^* \cdot \phi = 1$$

With this we interpret $|\psi|^2$ resp. $|\phi|^2$ as probability densities.

The position operator $\hat{x}$ multiplies with the position x: $\hat{x}(\psi(x)) = x \cdot \psi(x)$ In this sense ψ is an eigenfunction of the operator $\hat{x}$ with eigenvalue x.	In position space the momentum operator $\hat{p}$ differentiates with respect to position x: $\hat{p}(\psi) = -i \cdot \frac{\partial}{dx}\psi$ As we work with exponential functions the derivative always contains the original function itself. In this sense Ψ is an eigenfunction of the operator $\hat{p}$. The eigenvalue is part of the exponent of the original function.
	In momentum space the momentum operator multiplies with the p: $\hat{p}(\phi) = p\phi = \hbar k\phi$

The expectation value of the position operator $\langle\hat{x}\rangle$: $$\langle\hat{x}\rangle = \langle\psi\|\hat{x}\psi\rangle = \int_{-\infty}^{\infty} \psi^* \cdot x \cdot \psi = \int_{-\infty}^{\infty} x \cdot \|\psi\|^2$$ This is the center of the probability distribution. Note: x commutes with ψ.	The expectation value of the momentum operator $\langle\hat{p}\rangle$ we get by using Fourier: $$\langle\hat{p}\rangle = \langle\psi\|\hat{p}\psi\rangle = \langle\phi\|\hbar k\phi\rangle = \int_{-\infty}^{\infty} \hbar k \cdot \|\phi\|^2$$ This is the center of the probability distribution. Note: $\hbar$ and k commute with ϕ.

We know that the operators $\hat{x}$ and $\hat{p}$ are Hermitian so that they can change positions within the bra-kets.

$$\langle\hat{x}^2\rangle = \langle\psi|\hat{x}^2\psi\rangle = \langle\hat{x}\psi|\hat{x}\psi\rangle$$

$$\langle\hat{p}^2\rangle = \langle\psi|p^2\psi\rangle = \langle\hat{p}\psi|\hat{p}\psi\rangle$$

Cauchy-Schwarz inequality gives:

$$\langle\hat{x}\psi|\hat{x}\psi\rangle\langle\hat{p}\psi|\hat{p}\psi\rangle \geq |\langle\hat{x}\psi|\hat{p}\psi\rangle|^2 = |\langle\psi|\hat{x}\hat{p}\psi\rangle|^2 = |\langle\hat{x}\hat{p}\rangle|^2$$

We split $\hat{x}\hat{p}$:

$$\hat{x}\hat{p} = \frac{1}{2}\hat{x}\hat{p} + \frac{1}{2}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p} - \frac{1}{2}\hat{p}\hat{x}$$

$\frac{1}{2}\hat{x}\hat{p} - \frac{1}{2}\hat{p}\hat{x}$ is known as the commutator $[\hat{x},\hat{p}]$.

$\frac{1}{2}\hat{x}\hat{p} + \frac{1}{2}\hat{p}\hat{x}$ is known as the anticommutator $\{\hat{x},\hat{p}\}$.

We rewrite:

$$\hat{x}\hat{p} = \frac{1}{2}\{\hat{x},\hat{p}\} + \frac{1}{2}[\hat{x},\hat{p}]$$

The anticommutator is Hermitian and gives a real expectation value, $\langle\{\hat{x},\hat{p}\}\rangle \in \mathbb{R}$.

The commutator is anti-Hermitian and gives a pure imaginary expectation value,
$\langle[\hat{x},\hat{p}]\rangle = i \cdot a, a \in \mathbb{R}$.

We get:

$$|\langle\hat{x}\hat{p}\rangle| = \left|\frac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle + \frac{1}{2}\langle[\hat{x},\hat{p}]\rangle\right| \geq \frac{1}{2}|\langle[\hat{x},\hat{p}]\rangle|$$