This paper deals with substituting variables in functions with respect to derivation. We work with a polynomial and an exponential function.

Hope I can help you with learning quantum mechanics.

Case 1: We substitute within a polynomial

We use a polynomial function:

$$u(x) = x^2 + 3 \cdot x + 5$$

We get the derivatives:

$\frac{du(x)}{dx} = 2 \cdot x + 3$	$\frac{d^2u(x)}{dx^2} = 2$

We use a new variable y:

$$y(x) := a \cdot x^2$$

$$x(y) := \sqrt{\frac{y}{a}}$$

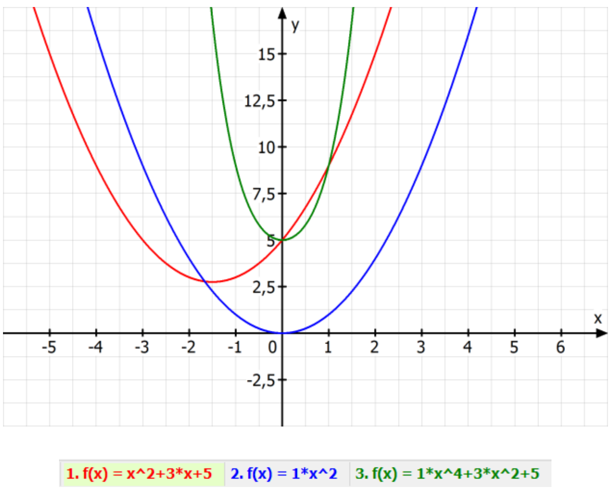
The derivatives:

$\frac{dy(x)}{dx} = 2 \cdot a \cdot x$	$\frac{d^2y(x)}{dx^2} = 2 \cdot a$

We get a new function:

$$h := u\big(y(x)\big) = a^2 \cdot x^4 + 3 \cdot a \cdot x^2 + 5$$

We plot this with $a := 1$:



We can look at h as a function of y:	We can look at h as a function of x:
$h(y) = y^2 + 3 \cdot y + 5$ Note: This is essentially the same function as $u(x)$. We renamed the variable x to y.	$h(x) = \big(y(x)\big)^2 + 3 * y(x) + 5 =$ $(a \cdot x^2)^2 + 3 \cdot a \cdot x^2 + 5 =$ $a^2 \cdot x^4 + 3 \cdot a \cdot x^2 + 5$ Note: This is a new function. We substituted the variable x by a function $y(x)$.

We get new derivatives.

$\frac{d\left(h(y)\right)}{dy} = 2 \cdot y + 3$	$\frac{d\left(h(x)\right)}{dx} = 4 \cdot a^2 \cdot x^3 + 6 \cdot a \cdot x$
$\frac{d^2\left(h(y)\right)}{dy^2} = 2$	$\frac{d^2\left(h(x)\right)}{dx^2} = 12 \cdot a^2 \cdot x^2 + 6 \cdot a =$

We can express the derivatives via the other variable.

$\frac{d\left(h(y)\right)}{dy} =$ $2 \cdot y(x) + 3 =$ $2 \cdot a \cdot x^2 + 3$	$\frac{d\left(h(x)\right)}{dx} =$ $4 \cdot a^2 \cdot x^3 + 6 \cdot a \cdot x =$ $\frac{1}{x} \cdot (4 \cdot a^2 \cdot x^4 + 6 \cdot a \cdot x^2) =$ $\frac{1}{x} \cdot (4 \cdot y^2 + 6 \cdot y) =$ $\sqrt{\frac{y}{a}} \cdot (4 \cdot y^2 + 6 \cdot y)$
$\frac{d^2\left(h(y)\right)}{dy^2} = 2$	$\frac{d^2\left(h(x)\right)}{dx^2} =$ $a \cdot (12 \cdot a \cdot x^2 + 6) =$ $a \cdot (12 \cdot y + 6)$

We can mix the derivatives of second order. We try:

$\frac{d^2}{dxdy}$	$\frac{d^2}{dydx}$
$\frac{d^2(h)}{dxdy} = \frac{d}{dy}\left(\frac{d(h)}{dx}\right) =$ $\frac{d}{dy}\left(\sqrt{\frac{a}{y}} \cdot (4 \cdot y^2 + 6 \cdot y)\right) =$ $\frac{d}{dy}\left(\sqrt{a} \cdot \left(4 \cdot y^{\frac{3}{2}} + 6 \cdot y^{\frac{1}{2}}\right)\right) =$ $\sqrt{a} \cdot \left(4 \cdot \frac{3}{2} \cdot y^{\frac{1}{2}} + 6 \cdot \frac{1}{2} \cdot y^{-\frac{1}{2}}\right) =$ $\sqrt{a} \cdot \left(6 \cdot y^{\frac{1}{2}} + 3 \cdot y^{-\frac{1}{2}}\right) =$ $\sqrt{\frac{a}{y}} \cdot (6 \cdot y + 3)$	$\frac{d^2(h)}{dydx} = \frac{d}{dx}\left(\frac{d(h)}{dy}\right) =$ $\frac{d}{dx}(2 \cdot a \cdot x^2 + 3) =$ $4 \cdot a \cdot x$
We can express y by x: $y = a \cdot x^2$	We can express x by y: $x = \sqrt{\frac{y}{a}}$
$\frac{d^2(h)}{dxdy} = \sqrt{\frac{a}{y}}(6 \cdot y + 3) =$ $\frac{1}{x} \cdot (6 \cdot a \cdot x^2 + 3)$	$\frac{d^2(h)}{dydx} = 4 \cdot \sqrt{ay}$

We check the chain rule:

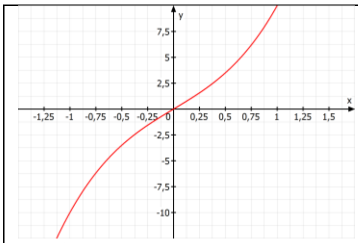
$\frac{d(h)}{dx} = \frac{d(h)}{dy} \cdot \frac{d(y)}{dx} =$ $(2 \cdot y + 3) \cdot (2 \cdot a \cdot x) =$ $(2 \cdot a \cdot x^2 + 3) \cdot (2 \cdot a \cdot x) =$ $4 \cdot a^2 \cdot x^3 + 6 \cdot a \cdot x$	$\frac{d(h)}{dx} = 4 \cdot a^2 \cdot x^3 + 6 \cdot a \cdot x$
$\frac{d(h)}{dy} = \frac{d(h)}{dx} \cdot \frac{dx}{dy} =$ $(4 \cdot a^2 \cdot x^3 + 6 \cdot a \cdot x) \cdot \frac{1}{2 \cdot a \cdot x} =$ $2 \cdot a \cdot x^2 + 3 =$ $2 \cdot y + 3$	$\frac{d(h)}{dy} = 2 \cdot y + 3$

We used:

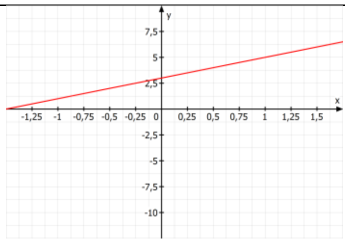
$$x = \sqrt{\frac{y}{a}} = \left(\frac{y}{a}\right)^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{1}{2a}\left(\frac{y}{a}\right)^{-\frac{1}{2}} = \frac{1}{2 \cdot a}(x^2)^{-\frac{1}{2}} = \frac{1}{2 \cdot a \cdot x}$$

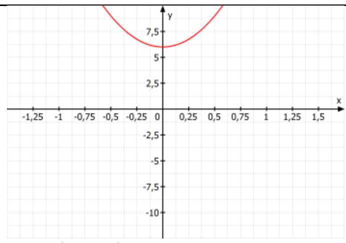
We plot this, using $a = 1$.



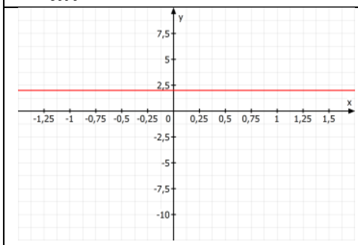
$$\frac{d(h)}{dx} = 4 \cdot a^2 \cdot x^3 + 6 \cdot a \cdot x$$



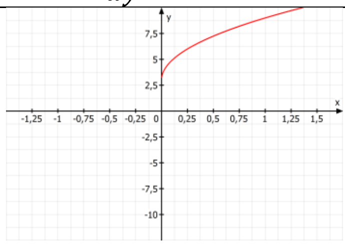
$$\frac{d(h)}{dy} = 2 \cdot y + 3$$



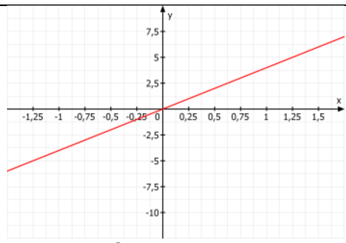
$$\frac{d^2(h(x))}{dx^2} = 12 \cdot a^2 \cdot x^2 + 6 \cdot a$$



$$\frac{d^2(h(y))}{dy^2} = 2$$



$$\frac{d^2(h)}{dxdy} = \sqrt{\frac{a}{y}} \cdot (6 \cdot y + 3)$$



$$\frac{d^2(h)}{dydx} = 4 \cdot a \cdot x$$

Case 2: We substitute within an exponential

We use an exponential function:

$$u(x) = x \cdot e^x$$

We get the derivatives:

$\frac{du(x)}{dx} = e^x + x \cdot e^x = (1+x) \cdot e^x$	$\frac{d^2u(x)}{dx^2} = x \cdot e^x + (1+x) \cdot e^x = (1 + 2 \cdot x) \cdot e^x$

We use a new variable y:

$$y(x) = a \cdot x^2$$

The derivatives:

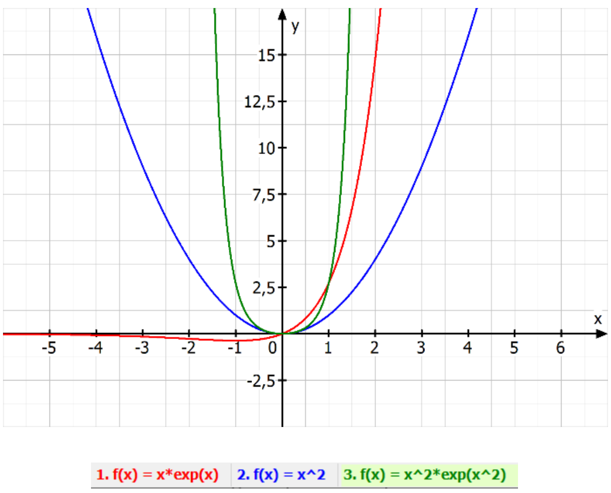
$\frac{dy(x)}{dx} = 2 \cdot a \cdot x$	$\frac{d^2y(x)}{dx^2} = 2 \cdot a$

We get a new function:

$$h \coloneqq u\big(y(x)\big) = a \cdot x^2 \cdot e^{a \cdot x^2}$$

The function h becomes ambiguous, we can express it either as a function of x or a function of y.

We plot this with $a \coloneqq 1$:



We get new function values.

We can look at h as a function of y:	We can look at h as a function of x:
$h(y) = y \cdot e^y$ Note: This is essentially the same function as $u(x)$. We renamed the variable x to y.	$h(x) = a \cdot x^2 \cdot e^{a \cdot x^2}$ Note: This is a new function compared to $u(x)$. We substituted the variable x by a function $y(x)$.

We get new derivatives.

$\frac{d(h(y))}{dy} = (1+y)\cdot e^y$	$\frac{d(h(x))}{dx} =$ $2\cdot a\cdot x\cdot e^{a\cdot x^2} + a\cdot x^2\cdot 2\cdot a\cdot x\cdot e^{a\cdot x^2} =$ $2\cdot a\cdot x\cdot e^{a\cdot x^2} + 2\cdot a^2\cdot x^3\cdot e^{a\cdot x^2} =$ $(2\cdot a\cdot x + 2\cdot a^2\cdot x^3)\cdot e^{a\cdot x^2}$
$\frac{d^2(h(y))}{dy^2} = \frac{d}{dy}\left(\frac{d(h(y))}{dy}\right) =$ $\frac{d}{dy}((1+y)\cdot e^y) =$ $e^y + (1+y)\cdot e^y =$ $(2+y)\cdot e^y$	$\frac{d^2(h(x))}{dx^2} = \frac{d}{dx}\left(\frac{d(h(x))}{dx}\right)$ $\frac{d}{dx}\left((2\cdot a\cdot x + 2\cdot a^2\cdot x^3)\cdot e^{a\cdot x^2}\right) =$ $(2\cdot a + 6\cdot a^2\cdot x^2)\cdot e^{a\cdot x^2} + (2\cdot a\cdot x + 2\cdot a^2\cdot x^3)\cdot 2\cdot a\cdot x\cdot e^{a\cdot x^2} =$ $(2\cdot a + 6\cdot a^2\cdot x^2 + 4\cdot a^2\cdot x^2 + 4\cdot a^3\cdot x^4)\cdot e^{a\cdot x^2} =$ $(2\cdot a + 10\cdot a^2\cdot x^2 + 4\cdot a^3\cdot x^4)\cdot e^{a\cdot x^2} =$ $a\cdot(2 + 10\cdot a\cdot x^2 + 4\cdot a^2\cdot x^4)\cdot e^{a\cdot x^2}$

We can express the derivatives via the other variable.

$\frac{d(h(y))}{dy} = (1+y(x))\cdot e^{y(x)} =$ $(1+a\cdot x^2)\cdot e^{a\cdot x^2}$	$\frac{d(h(x))}{dx} = (2\cdot a\cdot x + 2\cdot a^2\cdot x^3)\cdot e^{a\cdot x^2} =$ $\frac{1}{x}\cdot(2\cdot a\cdot x^2 + 2\cdot a^2\cdot x^4)\cdot e^{a\cdot x^2} =$ $2\cdot\sqrt{\frac{a}{y}}\cdot(y+y^2)\cdot e^y$
$\frac{d^2(h(y))}{dy^2} = (2+y)\cdot e^y =$ $(2+a\cdot x^2)\cdot e^{a\cdot x^2}$	$\frac{d^2(h(x))}{dx^2} =$ $a\cdot(2 + 10\cdot a\cdot x^2 + 4\cdot a^2\cdot x^4)\cdot e^{a\cdot x^2} =$ $a\cdot(2 + 10\cdot y + 4\cdot y^2)\cdot e^y$

We can mix the derivatives of second order. We try:

$\frac{d^2}{dxdy}$	$\frac{d^2}{dydx}$
$\frac{d^2(h)}{dxdy} = \frac{d}{dy}\left(\frac{d(h)}{dx}\right) =$ $\frac{d}{dy}\left(2\cdot\sqrt{\frac{a}{y}}\cdot(y+y^2)\cdot e^y\right) =$ $\frac{d}{dy}\left(2\cdot\sqrt{a}\cdot y^{-\frac{1}{2}}(y+y^2)\cdot e^y\right) =$ $\frac{d}{dy}\left(2\cdot\sqrt{a}\cdot\left(y^{\frac{1}{2}} + y^{\frac{3}{2}}\right)\cdot e^y\right) =$ $2\cdot\sqrt{a}\cdot\left(\frac{1}{2}y^{-\frac{1}{2}} + \frac{3}{2}y^{\frac{1}{2}}\right)\cdot e^y + 2\cdot\sqrt{a}\cdot\left(y^{\frac{1}{2}} + y^{\frac{3}{2}}\right)\cdot e^y =$ $2\cdot\sqrt{a}\cdot\left(\frac{1}{2}y^{-\frac{1}{2}} + \frac{3}{2}y^{\frac{1}{2}} + y^{\frac{1}{2}} + y^{\frac{3}{2}}\right)\cdot e^y =$	$\frac{d^2(h)}{dydx} =$ $\frac{d}{dx}\left(\frac{d(h)}{dy}\right) =$ $\frac{d}{dx}\left((1+a\cdot x^2)\cdot e^{a\cdot x^2}\right) =$ $2\cdot a\cdot e^{a\cdot x^2} + (1+a\cdot x^2)\cdot 2\cdot a\cdot e^{a\cdot x^2} =$ $(2\cdot a + (1+a\cdot x^2)\cdot 2\cdot a)\cdot e^{a\cdot x^2} =$ $(2\cdot a + 2\cdot a + 2\cdot a^2\cdot x^2)\cdot e^{a\cdot x^2} =$ $2\cdot a\cdot(2+a\cdot x^2)\cdot e^{a\cdot x^2}$

$2\cdot\sqrt{a}\cdot\left(\frac{1}{2}y^{-\frac{1}{2}}+\frac{5}{2}y^{\frac{1}{2}}+y^{\frac{3}{2}}\right)\cdot e^{y}=$ $2\cdot\sqrt{a}\cdot\frac{1}{2}y^{-\frac{1}{2}}\cdot(1+5\cdot y+2\cdot y^{2})\cdot e^{y}=$ $\sqrt{\frac{a}{y}}\cdot(1+5\cdot y+2\cdot y^{2})\cdot e^{y}$	
We can express x by y: $y=a\cdot x^{2}\rightarrow x=\sqrt{\frac{y}{a}}$	We can express y by x: $y=a\cdot x^{2}$
$\frac{d^{2}(h)}{dxdy}=\sqrt{\frac{a}{y}}\cdot(1+5\cdot y+2\cdot y^{2})\cdot e^{y}=$ $\frac{1}{x}\cdot(1+5\cdot a\cdot x^{2}+2\cdot a^{2}\cdot x^{4})\cdot e^{a\cdot x^{2}}$	$\frac{d^{2}(h)}{dydx}=2\cdot a\cdot(2+a\cdot x^{2})\cdot e^{a\cdot x^{2}}=$ $2\cdot a\cdot(2+y)\cdot e^{y}$

We check the chain rule:

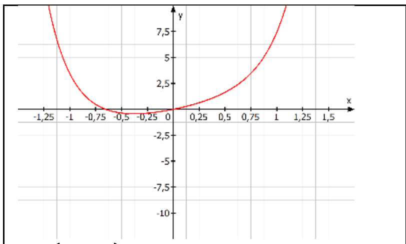
$\frac{d(h)}{dx}=\frac{d(h)}{dy}\cdot\frac{d(y)}{dx}=$ $(1+y)\cdot e^{y}\cdot(2\cdot a\cdot x)=$ $(1+a\cdot x^{2})\cdot e^{a\cdot x^{2}}\cdot(2\cdot a\cdot x)=$ $(2\cdot a\cdot x+2\cdot a^{2}\cdot x^{3})\cdot e^{a\cdot x^{2}}$	$\frac{d(h)}{dx}=(2\cdot a\cdot x+2\cdot a^{2}\cdot x^{3})\cdot e^{a\cdot x^{2}}$
$\frac{d(h)}{dy}=\frac{d(h)}{dx}\cdot\frac{dx}{dy}=$ $(2\cdot a\cdot x+2\cdot a^{2}\cdot x^{3})\cdot e^{a\cdot x^{2}}\cdot\frac{1}{2\cdot a\cdot x}=$ $(1+a\cdot x^{2})\cdot e^{a\cdot x^{2}}=$ $(1+y)\cdot e^{y}$	$\frac{d(h)}{dy}=(1+y)\cdot e^{y}$

We used:

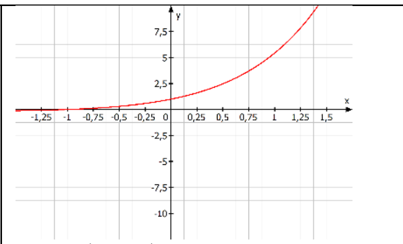
$$x=\sqrt{\frac{y}{a}}=\left(\frac{y}{a}\right)^{\frac{1}{2}}$$

$$\frac{dx}{dy}=\frac{1}{2a}\left(\frac{y}{a}\right)^{-\frac{1}{2}}=\frac{1}{2\cdot a}(x^{2})^{-\frac{1}{2}}=\frac{1}{2\cdot a\cdot x}$$

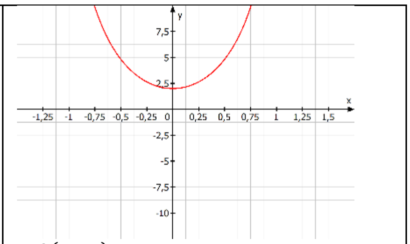
We plot this, using $a = 1$.



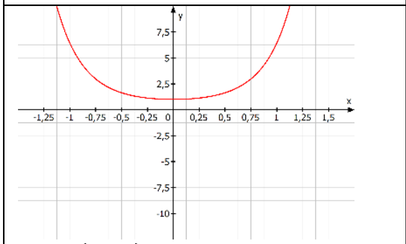
$$\frac{d\big(h(x)\big)}{dx} = (2ax + 2a^2x^3)e^{ax^2}$$



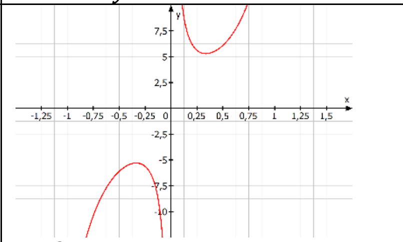
$$\frac{d\big(h(y)\big)}{dy} = (1 + y) \cdot e^{y}$$



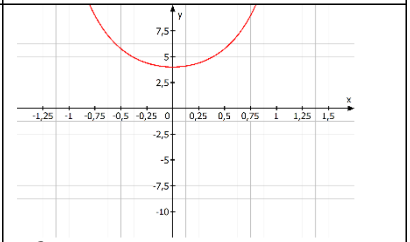
$$\frac{d^2\big(h(x)\big)}{dx^2} = a(2 + 10ax^2 + 4a^2x^4)e^{ax^2}$$



$$\frac{d^2\big(h(y)\big)}{dy^2} = (2 + a \cdot x^2) \cdot e^{a \cdot x^2}$$



$$\frac{d^2}{dxdy} = \frac{1}{x}(1 + 5ax^2 + 2a^2x^4)e^{ax^2}$$



$$\frac{d^2(h)}{dydx} = 2 \cdot a \cdot (2 + a \cdot x^2) \cdot e^{a \cdot x^2}$$