This short paper calculates (in a rough way) the speed of an electron in the hydrogen atom, assuming the hydrogen atom being in the ground state. We use the quantum mechanical virial theorem.

More Information about the virial theorem you find at:

https://www.physics.utah.edu/~lebohec/ScaleRelativity/Virial/virial.pdf

Hope I can help you with learning quantum mechanics.

The Hamiltonian:

$$H = \frac{\vec{p}^2}{2m} + V(\vec{r})$$

Note: we work with a central potential $V(\vec{r})$ and omit vector arrows and operator hats ˆ.

The quantum mechanical Virial theorem gives the kinetic energy $\langle T \rangle$ for a particle in a central potential:

$$\langle T \rangle = \langle \frac{1}{2} r \frac{\partial V}{\partial r} \rangle$$

Note: expectation values are constant for stationary states only.

The kinetic operator:

$$T = \frac{p^2}{2m}$$

The potential $\sim \frac{1}{r}$:

$$V(r) = \lambda \frac{1}{r}$$

$$\frac{\partial V}{\partial r} = -\frac{\lambda}{r^2}$$

From the Virial theorem we get the kinetic energy:

$$\langle T \rangle = \langle -\frac{\lambda}{2r} \rangle = -\frac{1}{2} \langle V \rangle$$

We have the relation:

$$\langle V \rangle = -2 \langle T \rangle$$

With this we get the Hamiltonian (the total energy):

$$\langle H \rangle = \langle T \rangle + \langle V \rangle \rightarrow$$

$$\langle H \rangle = -\langle T \rangle$$

The Hamiltonian of the hydrogen atom:

$$H = \frac{p^2}{2m_e} - \frac{e^2}{4\pi\varepsilon_0 r}$$

We have the kinetic energy:

$$\frac{p^2}{2m_e}$$

We have the potential energy:

$$\frac{e^2}{4\pi\varepsilon_0 r}$$

We use the Bohr radius a_o:

$$a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2}$$

The ground state energy of the hydrogen atom:

$$E_{gs} = -\frac{m_e e^4}{2 \cdot (4\pi\varepsilon_0)^2 \hbar^2} = -\frac{1}{2} m_e c^2 \cdot \frac{e^4}{(4\pi\varepsilon_0 c)^2 \hbar^2} =$$

$$-\frac{1}{2} m_e c^2 \cdot \left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^2$$

Note: c is the speed of light.

We have the rest energy of the electron:

$$\frac{1}{2} m_e c^2$$

The square of the fine structure constant α:

$$\left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^2$$

We rewrite the ground state energy:

$$E_{gs} = -\frac{1}{2} m_e c^2 \cdot \alpha^2$$

Note: the energy of the ground state is negative, it is a bound state.

With this we can answer the following question: what is the expectation value of the speed of the electron in the ground state of the hydrogen atom, $\langle v^2 \rangle$?

We have no operator for speed in quantum mechanics, so we build one:

$$\langle v^2 \rangle = \frac{1}{{m_e}^2} \langle p^2 \rangle$$

We use the Virial theorem to calculate the expectation value of the Hamiltonian:

$$\langle H \rangle = -\langle T \rangle = -\frac{\langle p^2 \rangle}{2m_e} = -\frac{1}{2} m_e \langle v^2 \rangle$$

On the other hand, the expectation value of the Hamiltonian is the ground state energy:

$$\langle H \rangle = E_{gs} \rightarrow$$

$$-\frac{1}{2} m_e \langle v^2 \rangle = -\frac{1}{2} m_e c^2 \alpha^2$$

$$\langle v^2 \rangle = c^2 \alpha^2$$

Conclusion:

We set $\langle v^2 \rangle \sim v^2$ and get:

$$v \sim \frac{c}{137}$$

This is a nonrelativistic speed.