Wave functions, operators and infinite dimensional Hilbert spaces.

Related information you may find at:

Griffiths, Chapter 3, Formalism

https://web.stanford.edu/class/rad226a/Lectures/Lecture4-2017-Quantum-II.pdf

Tutorium Quantenmechanik, J.-M. Schwindt, Springer, ISBN 978-3-642-37791-4. (… written in German…)

Related topics you find in the papers “Continuous functions as vectors” and “Discrete probability and probability density” on this website.

Hope I can help you with learning quantum mechanics.

Contents

Vector space

Let V be a vector space over the field $\mathbb{C}$.

Let M be a set.

The set of functions $F(M, V) = \{f: M \rightarrow V\}$ will also represent a vector space over $\mathbb{C}$. We call it space of functions.

Note: $\mathbb{R} \subset \mathbb{C}$

We are in search for position operator X and differentiation operator D.

The position operator X multiplies any function by x:

$$Xf(x) = xf(x)$$

The differentiation operator D differentiates any function with respect to x:

$$Df(x) = \frac{d}{dx} f(x)$$

If we work with matrices as operators, we have the eigenvalue/eigenvector behavior:

$$Xf(x) = \lambda f(x)$$

$$Df(x) = \lambda f(x)$$

Note: $f(x)$ should be a vector in this case.

Scalar product

In the space of functions, we define a dot-product.

Let f, g be functions:

$$\langle f|g \rangle = \int_x f^*(r) g(r) dx$$

Note: We work in one dimension only.

Note: $f^*(r)$ denotes complex conjugation.

Properties of the dot-product:

- antilinear in the first argument, $\langle af + bg|h \rangle = a^* \langle f|h \rangle + b^* \langle g|h \rangle$
- linear in the second argument, $\langle f|ag + bh \rangle = a \langle f|g \rangle + b \langle f|h \rangle$
- Hermitian, $\langle f|g \rangle = \langle g|f \rangle^*$
- positive definite, $\langle f|f \rangle \geq 0$, $\langle f|f \rangle = 0 \leftrightarrow |f\rangle = \vec{0}$

We check the properties:

Antilinear in the first argument:	$\langle af + bg \| h \rangle =$ $\int_x \left(a^* f^*(r) + b^* g^*(r)\right) h(r) dx =$ $a^* \int_x f^*(r) h(r) dx + b^* \int_x g^*(r) h(r) dx =$ $a^* \langle f \| h \rangle + b^* \langle g \| h \rangle$
Linear in the second argument:	$\langle f \| ag + bh \rangle =$ $\int_x f^*(r) \left(ag(r) + bh(r)\right) dx =$ $\int_x f^*(r) a g(r) dx + \int_x f^*(r) b h(r) dx =$ $a \int_x f^*(r) g(r) dx + b \int_x f^*(r) h(r) dx =$ $a\langle f \| g \rangle + b \langle f \| h \rangle$
Hermitian:	$\langle f \| g \rangle = \langle f \| g \rangle^{**} = \langle f^* \| g^* \rangle^* =$ $\left(\int_x f^{**}(r) g^*(r) dx \right)^* = \left(\int_x f(r) g^*(r) dx \right)^* =$ $\left(\int_x g^*(r) f(r) dx \right)^* = \langle g \| f \rangle^*$
Positive definite:	$\langle f \| f \rangle = \int_x f^*(r) f(r) dx =$ $\int_x \|f(r)\|^2 d^n x \geq 0$ $\|f(r)\|^2 = 0 \leftrightarrow f(r) = 0$

From the dot-product we get a norm:

$$\|f\| = \sqrt{\int_x f^*(r) g(r) dx}$$

Quantum mechanics constraints to functions the integral over x remains finite, the so-called square-integrable functions or L^2-functions. For these functions holds the necessary condition:

$$\lim_{x \to \infty} f(r) = 0$$

What we have so far is a Hilbert vector space of square-integrable functions:

$$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$$

In this space are valid the Schwarz inequality and the triangle inequality:

$$|\langle f|g\rangle| \leq \|f\| \cdot \|g\|$$

$$\|f+g\| \leq \|f\| + \|g\|$$

Orthonormal basis

Polynomials are good-natured functions that easily can be broken down to a sum of basis-functions:

$$P(x) = c_0 \cdot x^0 + c_1 \cdot x^1 + \cdots + c_n \cdot x^n$$

Basis vectors are:

$$e_0 = x^0, e_1 = x^1, e_2 = x^2, \ldots$$

Note: The basis-functions are called monomials.
Note: All polynomials are built from these basis functions.

Polynomials are not square-integrable functions:

$$\int_{-\infty}^{\infty} P(x)dx = \infty$$

Note: with the exception of the zero polynomial.

We use the exponential $e^{-\frac{x^2}{2}}$ to get the property "square-integrable":

$$e^{-\frac{x^2}{2}} \cdot P(x)$$

We get basis functions:

$$e_0(x) = e^{-\frac{x^2}{2}}, \qquad e_1(x) = e^{-\frac{x^2}{2}} \cdot x, \qquad e_2(x) = e^{-\frac{x^2}{2}} \cdot x^2, \ldots, e_i(x) = e^{-\frac{x^2}{2}} \cdot x^i$$

These are linearly independent and can serve as a basis.

We will orthonormalize them via the Gram-Schmidt procedure.

Note: The Gram-Schmidt procedure you find e. g. in Wikipedia.

We start with e_0:

$$\tilde{b}_0 \coloneqq e_0(x) = e^{-\frac{x^2}{2}} \cdot x^0$$

We normalize:

$$\sqrt{\langle \tilde{b}_0 | \tilde{b}_0 \rangle} = \sqrt{\int_{-\infty}^{\infty} e_0(x) e_0(x) dx} = \sqrt{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{x^2}{2}} dx} = \sqrt{\int_{-\infty}^{\infty} e^{-x^2} dx} = \sqrt{\sqrt{\pi}} = \pi^{\frac{1}{4}}$$

We get our first normalized basis function:

$$b_0 := \frac{e^{-\frac{x^2}{2}}}{\pi^{\frac{1}{4}}} = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$$

Note: $\|b_0\| = \sqrt{\langle b_0|b_0\rangle} = 1$.

Next one, basis-function two, b_1:

We orthogonalize:

$$\tilde{b}_1 := e_1 - \frac{\langle b_0|e_1\rangle}{\langle b_0|b_0\rangle} b_0 \rightarrow$$

$$\tilde{b}_1 = e^{-\frac{x^2}{2}} \cdot x - \frac{\int_{-\infty}^{\infty} \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{x^2}{2}} \cdot x dx}{1} \cdot \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} =$$

$$e^{-\frac{x^2}{2}} \cdot x - \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot x dx \cdot e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} \cdot x - 0 =$$

$$e^{-\frac{x^2}{2}} \cdot x$$

We normalize:

$$\sqrt{\langle \tilde{b}_1|\tilde{b}_1\rangle} = \sqrt{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot x e^{-\frac{x^2}{2}} \cdot x dx} = \sqrt{\int_{-\infty}^{\infty} x^2 e^{-x^2} dx} = \sqrt{\frac{1}{2}\pi^{\frac{1}{2}}} = \sqrt{\frac{1}{2}} \cdot \pi^{\frac{1}{4}}$$

We get the second orthonormal basis function:

$$b_1 = \frac{e^{-\frac{x^2}{2}} \cdot x}{\sqrt{\frac{1}{2}}\pi^{\frac{1}{4}}} = \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \cdot x$$

Next one, basis function three, b_2. We note that basis functions are normalized:

$$\langle b_0|b_0\rangle = \langle b_1|b_1\rangle = 1$$

We calculate:

$$\tilde{b}_2 := e_2 - \frac{\langle b_0|e_2\rangle}{\langle b_0|b_0\rangle} b_0 - \frac{\langle b_1|e_2\rangle}{\langle b_1|b_1\rangle} b_1 =$$

$$e_2 - \langle b_0|e_2\rangle b_0 - \langle b_1|e_2\rangle b_1 \rightarrow$$

$$\tilde{b}_2 := e^{-\frac{x^2}{2}} \cdot x^2 - \int_{-\infty}^{\infty} \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{x^2}{2}} x^2 dx\, \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} - \int_{-\infty}^{\infty} \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} x \cdot e^{-\frac{x^2}{2}} x^2 dx\, \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} x =$$

$$e^{-\frac{x^2}{2}} \cdot x^2 - \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot x^2 dx \cdot e^{-\frac{x^2}{2}} - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot x^3 dx \cdot e^{-\frac{x^2}{2}} \cdot x =$$

$$e^{-\frac{x^2}{2}} \cdot x^2 - \pi^{-\frac{1}{2}} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot e^{-\frac{x^2}{2}} - 0 =$$

$$e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right)$$

We normalize:

$$\left\| e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) \right\| =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) \cdot e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) dx} = \sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^2 - \frac{1}{2}\right)^2 dx} =$$

$$\sqrt{\int_{-\infty}^{\infty} x^4 e^{-x^2} dx - \int_{-\infty}^{\infty} x^2 e^{-x^2} dx + \frac{1}{4} \int_{-\infty}^{\infty} e^{-x^2} dx} =$$

$$\sqrt{\frac{3}{4}\sqrt{\pi} - \frac{1}{2}\sqrt{\pi} + \frac{1}{4}\sqrt{\pi}} = \pi^{\frac{1}{4}} \sqrt{\frac{3 - 2 + 1}{4}} = \pi^{\frac{1}{4}} \sqrt{\frac{1}{2}}$$

The third orthonormal basis function b_2:

$$b_2 = \frac{e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right)}{\pi^{\frac{1}{4}} \sqrt{\frac{1}{2}}} = \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right)$$

Next one, basis function four, b_3. We note that basis functions are normalized:

$$\langle b_0 | b_0 \rangle = \langle b_1 | b_1 \rangle = \langle b_2 | b_2 \rangle = 1$$

We calculate:

$$\tilde{b}_3 := e_3 - \frac{\langle b_0 | e_3 \rangle}{\langle b_0 | b_0 \rangle} b_0 - \frac{\langle b_1 | e_3 \rangle}{\langle b_1 | b_1 \rangle} b_1 - \frac{\langle b_2 | e_3 \rangle}{\langle b_2 | b_2 \rangle} b_2 =$$

$$e_3 - \langle b_0 | e_3 \rangle b_0 - \langle b_1 | e_3 \rangle b_1 - \langle b_2 | e_3 \rangle b_2 \rightarrow$$

$$\tilde{b}_3 := e^{-\frac{x^2}{2}} x^3 - \int_{-\infty}^{\infty} \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{x^2}{2}} x^3 dx\, \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} - \int_{-\infty}^{\infty} \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} x \cdot e^{-\frac{x^2}{2}} x^3 dx\, \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} x$$

$$- \int_{-\infty}^{\infty} \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) \cdot e^{-\frac{x^2}{2}} x^3 dx \sqrt{2}\pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) =$$

$$e^{-\frac{x^2}{2}} x^3 - \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} x^3 dx\, e^{-\frac{x^2}{2}} - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} x^4 dx\, e^{-\frac{x^2}{2}} x$$

$$- 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \left(x^2 - \frac{1}{2}\right) x^3 dx e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) =$$

$$e^{-\frac{x^2}{2}} x^3 - 0 - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} x^4 dx\, e^{-\frac{x^2}{2}} x - 0 =$$

$$e^{-\frac{x^2}{2}} x^3 - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} x^4 dx\, e^{-\frac{x^2}{2}} x =$$

$$e^{-\frac{x^2}{2}}x^3 - 2\pi^{-\frac{1}{2}}\frac{3}{4}\sqrt{\pi}e^{-\frac{x^2}{2}}x =$$

$$e^{-\frac{x^2}{2}}x^3 - \frac{3}{2}e^{-\frac{x^2}{2}}x =$$

$$e^{-\frac{x^2}{2}}\left(x^3 - \frac{3}{2}x\right)$$

We normalize:

$$\left\|e^{-\frac{x^2}{2}}\left(x^3 - \frac{3}{2}x\right)\right\| =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) \cdot e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) dx} = \sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^3 - \frac{3}{2}x\right)^2 dx} =$$

$$\sqrt{\int_{-\infty}^{\infty} x^6 e^{-x^2} dx - 3\int_{-\infty}^{\infty} x^4 e^{-x^2} dx + \frac{9}{4}\int_{-\infty}^{\infty} x^2 e^{-x^2} dx} =$$

$$\sqrt{\frac{15}{8}\sqrt{\pi} - \frac{9}{4}\sqrt{\pi} + \frac{9}{8}\sqrt{\pi}} = \pi^{\frac{1}{4}}\sqrt{\frac{15 - 18 + 9}{8}} =$$

$$\pi^{\frac{1}{4}}\sqrt{\frac{3}{4}}$$

The fourth orthonormal basis function b_3:

$$b_3 = \frac{e^{-\frac{x^2}{2}}\left(x^3 - \frac{3}{2}x\right)}{\pi^{\frac{1}{4}}\sqrt{\frac{3}{4}}} = \sqrt{\frac{4}{3}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right)$$

Next one, basis function five, b_4. We note that basis functions are normalized:

$$\langle b_0|b_0\rangle = \langle b_1|b_1\rangle = \langle b_2|b_2\rangle = \langle b_3|b_3\rangle = 1$$

We calculate:

$$\tilde{b}_4 := e_4 - \frac{\langle b_0|e_4\rangle}{\langle b_0|b_0\rangle}b_0 - \frac{\langle b_1|e_4\rangle}{\langle b_1|b_1\rangle}b_1 - \frac{\langle b_2|e_4\rangle}{\langle b_2|b_2\rangle}b_2 - \frac{\langle b_3|e_4\rangle}{\langle b_3|b_3\rangle}b_3 =$$

$$e_3 - \langle b_0|e_4\rangle b_0 - \langle b_1|e_4\rangle b_1 - \langle b_2|e_4\rangle b_2 - \langle b_3|e_4\rangle b_3 \rightarrow$$

$$\tilde{b}_4 := e^{-\frac{x^2}{2}}x^4 - \int_{-\infty}^{\infty} \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot e^{-\frac{x^2}{2}}x^4 dx\, \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} - \int_{-\infty}^{\infty} \sqrt{2}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}x \cdot e^{-\frac{x^2}{2}}x^4 dx\, \sqrt{2}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}x$$

$$- \int_{-\infty}^{\infty} \sqrt{2}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) \cdot e^{-\frac{x^2}{2}}x^4 dx\, \sqrt{2}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right)$$

$$- \int_{-\infty}^{\infty} \sqrt{\frac{4}{3}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) \cdot e^{-\frac{x^2}{2}}x^4 dx\, \sqrt{\frac{4}{3}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) =$$

$$e^{-\frac{x^2}{2}}x^4 - \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^4 dx\, e^{-\frac{x^2}{2}} - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^5 dx\, e^{-\frac{x^2}{2}} \cdot x$$

$$- 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^6 - \frac{1}{2}x^4\right) dx\, e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right)$$

$$- \frac{4}{3}\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^7 - \frac{3}{2}x^5\right) dx e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^4 - \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^4 dx - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^5 dx \cdot x - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^6 - \frac{1}{2}x^4\right) dx\right.$$

$$\left. \cdot \left(x^2 - \frac{1}{2}\right) - \frac{4}{3}\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^7 - \frac{3}{2}x^5\right) dx \cdot \left(x^3 - \frac{3}{2}x\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^4 - \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^4 dx - 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^5 dx \cdot x\right.$$

$$- 2\pi^{-\frac{1}{2}}\left(\int_{-\infty}^{\infty} e^{-x^2} \cdot x^6 dx - \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} \cdot x^4 dx\right) \cdot \left(x^2 - \frac{1}{2}\right)$$

$$\left. - \frac{4}{3}\pi^{-\frac{1}{2}}\left(\int_{-\infty}^{\infty} e^{-x^2} \cdot x^7 dx - \frac{4}{2}\int_{-\infty}^{\infty} e^{-x^2} \cdot x^5 dx\right) \cdot \left(x^3 - \frac{3}{2}x\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^4 - \pi^{-\frac{1}{2}}\frac{3}{4}\sqrt{\pi} - 0 - 2\pi^{-\frac{1}{2}}\left(\frac{15}{8}\sqrt{\pi} - \frac{1}{2}\frac{3}{4}\sqrt{\pi}\right) \cdot \left(x^2 - \frac{1}{2}\right) - 0\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^4 - \frac{3}{4} - 2\left(\frac{15}{8} - \frac{1}{2}\frac{3}{4}\right) \cdot \left(x^2 - \frac{1}{2}\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^4 - \frac{3}{4} - 3 \cdot \left(x^2 - \frac{1}{2}\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^4 - \frac{3}{4} - 3x^2 + \frac{3}{2}\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right) =$$

We normalize:

$$\left\|e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right)\right\| =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot \left(x^4 - 3x^2 + \frac{3}{4}\right) \cdot e^{-\frac{x^2}{2}} \cdot \left(x^4 - 3x^2 + \frac{3}{4}\right) dx} = \sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^4 - 3x^2 + \frac{3}{4}\right)^2 dx} =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^8 - 6x^6 + \frac{21}{2}x^4 - \frac{9}{2}x^2 + \frac{9}{16}\right) dx} =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot x^8 dx - 6 \int_{-\infty}^{\infty} e^{-x^2} \cdot x^6 dx + \frac{21}{2} \int_{-\infty}^{\infty} e^{-x^2} \cdot x^4 dx - \frac{9}{2} \int_{-\infty}^{\infty} e^{-x^2} \cdot x^2 dx + \frac{9}{16} \int_{-\infty}^{\infty} e^{-x^2} dx} =$$

$$\sqrt{\frac{105}{16}\sqrt{\pi} - \frac{90}{8}\sqrt{\pi} + \frac{63}{8}\sqrt{\pi} - \frac{9}{4}\sqrt{\pi} + \frac{9}{16}\sqrt{\pi}} = \pi^{\frac{1}{4}}\sqrt{\frac{105 - 180 + 126 - 36 + 9}{16}} =$$

$$\pi^{\frac{1}{4}}\sqrt{\frac{24}{16}} = \pi^{\frac{1}{4}}\sqrt{\frac{3}{2}}$$

The fifth orthonormal basis function b_4:

$$b_4 = \frac{e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right)}{\pi^{\frac{1}{4}}\sqrt{\frac{3}{2}}} = \sqrt{\frac{2}{3}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right)$$

Next and last one, basis function six, b_5. We note that basis functions are normalized:

$$\langle b_0|b_0\rangle = \langle b_1|b_1\rangle = \langle b_2|b_2\rangle = \langle b_3|b_3\rangle = \langle b_4|b_4\rangle = 1$$

We calculate:

$$\tilde{b}_5 := e_5 - \frac{\langle b_0|e_5\rangle}{\langle b_0|b_0\rangle}b_0 - \frac{\langle b_1|e_5\rangle}{\langle b_1|b_1\rangle}b_1 - \frac{\langle b_2|e_5\rangle}{\langle b_2|b_2\rangle}b_2 - \frac{\langle b_3|e_5\rangle}{\langle b_3|b_3\rangle}b_3 - \frac{\langle b_4|e_5\rangle}{\langle b_4|b_4\rangle}b_4 =$$

$$e_5 - \langle b_0|e_5\rangle b_0 - \langle b_1|e_5\rangle b_1 - \langle b_2|e_5\rangle b_2 - \langle b_3|e_5\rangle b_3 - \langle b_4|e_5\rangle b_4 \rightarrow$$

$$\tilde{b}_5 := e^{-\frac{x^2}{2}}x^5 - \int_{-\infty}^{\infty} \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}} \cdot e^{-\frac{x^2}{2}} \cdot x^5 dx \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}} - \int_{-\infty}^{\infty} \sqrt{2} \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}} \cdot x \cdot e^{-\frac{x^2}{2}} \cdot x^5 dx \cdot \sqrt{2}$$

$$\cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}} \cdot x - \int_{-\infty}^{\infty} \sqrt{2} \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) \cdot e^{-\frac{x^2}{2}} \cdot x^5 dx \cdot \sqrt{2} \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}}$$

$$\cdot \left(x^2 - \frac{1}{2}\right) - \int_{-\infty}^{\infty} \sqrt{\frac{4}{3}} \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) \cdot e^{-\frac{x^2}{2}} \cdot x^5 dx \cdot \sqrt{\frac{4}{3}} \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}}$$

$$\cdot \left(x^3 - \frac{3}{2}x\right) - \int_{-\infty}^{\infty} \sqrt{\frac{2}{3}} \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right) \cdot e^{-\frac{x^2}{2}} \cdot x^5 dx \cdot \sqrt{\frac{2}{3}} \cdot \pi^{-\frac{1}{4}}$$

$$\cdot e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right) =$$

$$e^{-\frac{x^2}{2}}x^5 - \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^5 dx \cdot e^{-\frac{x^2}{2}} - 2 \cdot \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^6 dx \cdot e^{-\frac{x^2}{2}}x$$
$$- 2\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^7 - \frac{1}{2}x^5\right) dx \cdot e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right)$$
$$- \frac{4}{3}\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^8 - \frac{3}{2}x^6\right) dx \cdot e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right)$$
$$- \frac{2}{3}\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \left(x^9 - 3x^7 + \frac{3}{4}x^5\right) dx \cdot e^{-\frac{x^2}{2}} \left(x^4 - 3x^2 + \frac{3}{4}\right) =$$

$$e^{-\frac{x^2}{2}}x^5 - 0 - 2 \cdot \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^6 dx\, e^{-\frac{x^2}{2}}x - 0 - \frac{4}{3}\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^8 - \frac{3}{2}x^6\right) dx e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right)$$
$$- 0 =$$

$$e^{-\frac{x^2}{2}}\left(x^5 - 2 \cdot \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}x^6 dx\, x - \frac{4}{3}\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^8 - \frac{3}{2}x^6\right) dx \cdot \left(x^3 - \frac{3}{2}x\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^5 - 2 \cdot \pi^{-\frac{1}{2}}\frac{15}{8}\sqrt{\pi}x - \frac{4}{3}\pi^{-\frac{1}{2}}\left(\frac{105}{16}\sqrt{\pi} - \frac{45}{16}\sqrt{\pi}\right)\left(x^3 - \frac{3}{2}x\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^5 - \frac{30}{8}x - \frac{4}{3}\left(\frac{60}{16}\right)\left(x^3 - \frac{3}{2}x\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^5 - \frac{30}{8}x - \frac{240}{48}\left(x^3 - \frac{3}{2}x\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^5 - \frac{30}{8}x - \frac{240}{48}x^3 + \frac{30}{4}x\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right)$$

We normalize:

$$\left\|e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right)\right\| =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right) \cdot e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right) dx} =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^5 - 5x^3 + \frac{15}{4}x\right)^2 dx} =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot \left(x^{10} - 10x^8 + \frac{65}{2}x^6 - \frac{75}{2}x^4 + \frac{225}{16}x^2\right) dx} =$$

$$\sqrt{\int_{-\infty}^{\infty} e^{-x^2} \cdot x^{10} dx - 10 \int_{-\infty}^{\infty} e^{-x^2} \cdot x^8 dx + \frac{65}{2} \int_{-\infty}^{\infty} e^{-x^2} \cdot x^6 dx - \frac{75}{2} \int_{-\infty}^{\infty} e^{-x^2} \cdot x^4 dx + \frac{225}{16} \int_{-\infty}^{\infty} e^{-x^2} \cdot x^2 dx}$$
$$=$$

$$\sqrt{\frac{945}{32}\sqrt{\pi} - 10\frac{105}{16}\sqrt{\pi} + \frac{65}{2}\frac{15}{8}\sqrt{\pi} - \frac{75}{2}\frac{3}{4}\sqrt{\pi} + \frac{225}{16}\frac{1}{2}\sqrt{\pi}} = \pi^{\frac{1}{4}}\sqrt{\frac{945}{32} - \frac{1050}{16} + \frac{975}{16} - \frac{225}{8} + \frac{225}{32}}$$
$$=$$

$$\pi^{\frac{1}{4}}\sqrt{\frac{945 - 2100 + 1950 - 900 + 225}{32}} =$$

$$\pi^{\frac{1}{4}}\sqrt{\frac{15}{4}}$$

The sixth orthonormal basis function b_5:

$$b_5 = \frac{e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right)}{\pi^{\frac{1}{4}}\sqrt{\frac{15}{4}}} = \sqrt{\frac{4}{15}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right)$$

We rewrite the six normalized basis-functions and compare with the Hermite polynomials:

$b_0(x) = \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}$	$H_0(x) = 1$
$b_1(x) = \sqrt{\frac{1}{2}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot 2x$	$H_1(x) = 2x$
$b_2(x) = \sqrt{\frac{1}{8}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot (4x^2 - 2)$	$H_2(x) = 4x^2 - 2$
$b_3(x) = \sqrt{\frac{1}{48}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}} \cdot (8x^3 - 12x)$	$H_3(x) = 8x^3 - 12x$
$b_4(x) = \sqrt{\frac{1}{384}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}(16x^4 - 48x^2 + 12)$	$H_4(x) = 16x^4 - 48x^2 + 12$
$b_5(x) = \sqrt{\frac{1}{3840}}\pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}(32x^5 - 160x^3 + 120x)$	$H_5(x) = 32x^5 - 160x^3 + 120x$

The orthonormal basis functions $b_i(x)$ correspond with the Hermite polynomials. The leading factors are result of the normalization process. The dependency becomes clearer if we work with the orthogonal, but not normalized basis functions $\tilde{b}_i(x)$:

$H_0(x) = 1$	$\tilde{b}_0 = e^{-\frac{x^2}{2}} \cdot 1$
$H_1(x) = 2x$	$\tilde{b}_1 = \frac{1}{2} e^{-\frac{x^2}{2}} \cdot 2x$
$H_2(x) = 4x^2 - 2$	$\tilde{b}_2 = \frac{1}{4} e^{-\frac{x^2}{2}} (4x^2 - 2)$
$H_3(x) = 8x^3 - 12x$	$\tilde{b}_3 = \frac{1}{8} e^{-\frac{x^2}{2}} (8x^3 - 12x)$
$H_4(x) = 16x^4 - 48x^2 + 12$	$\tilde{b}_4 = \frac{1}{16} e^{-\frac{x^2}{2}} (16x^4 - 48x^2 + 12)$
$H_5(x) = 32x^5 - 160x^3 + 120x$	$\tilde{b}_5 = \frac{1}{32} e^{-\frac{x^2}{2}} (32x^5 - 160x^3 + 120x)$

We can build the orthogonal basis functions $\tilde{b}_i$ by help of the Hermite polynomials $H_i(x)$:

$$\tilde{b}_i = \frac{1}{2^i} e^{-\frac{x^2}{2}} H_i(x)$$

This way we can proceed and build an infinite basis of orthonormal functions of the vector space of square-integrable polynomials without the recursive Gram-Schmidt process.

Transformation

We work with the orthogonal $\tilde{b}_i$-basis. We have two basis-functions for the functions space, the e_i-basis and the orthogonal $\tilde{b}_i$-basis:

$e_0(x) = e^{-\frac{x^2}{2}}$	$\tilde{b}_0(x) = e^{-\frac{x^2}{2}} \cdot 1$
$e_1(x) = e^{-\frac{x^2}{2}} \cdot x$	$\tilde{b}_1(x) = \frac{1}{2} e^{-\frac{x^2}{2}} \cdot 2x$
$e_2(x) = e^{-\frac{x^2}{2}} \cdot x^2$	$\tilde{b}_2(x) = \frac{1}{4} e^{-\frac{x^2}{2}} (4x^2 - 2)$
$e_3(x) = e^{-\frac{x^2}{2}} \cdot x^3$	$\tilde{b}_3(x) = \frac{1}{8} e^{-\frac{x^2}{2}} (8x^3 - 12x)$
$e_4(x) = e^{-\frac{x^2}{2}} \cdot x^4$	$\tilde{b}_4(x) = \frac{1}{16} e^{-\frac{x^2}{2}} (16x^4 - 48x^2 + 12)$
$e_5(x) = e^{-\frac{x^2}{2}} \cdot x^5$	$\tilde{b}_5(x) = \frac{1}{32} e^{-\frac{x^2}{2}} (32x^5 - 160x^3 + 120x)$

We can express the $\tilde{b}_i$ basis in terms of the e_i basis:

$$\tilde{b}_0(x) = e^{-\frac{x^2}{2}} = e_0(x)$$

$$\tilde{b}_1(x) = e^{-\frac{x^2}{2}} \cdot x = e_1(x)$$

$$\tilde{b}_2(x) = e^{-\frac{x^2}{2}} \left(x^2 - \frac{1}{2}\right) = e_2(x) - \frac{1}{2} e_0(x)$$

$$\tilde{b}_3(x) = e^{-\frac{x^2}{2}}\left(x^3 - \frac{3}{2}x\right) = e_3(x) - \frac{3}{2}e_1(x)$$

$$\tilde{b}_4(x) = e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right) = e_4(x) - 3e_2(x) + \frac{3}{4}e_0(x)$$

$$\tilde{b}_5(x) = e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}\right) = e_5(x) - 5e_3(x) + \frac{15}{4}e_1(x)$$

Analog to finite vector spaces we can build a transformation matrix T that changes the components of a vector $\tilde{b}_i$ into the components e_i.

We build the transformation matrix T:

$$T := \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{15}{4} \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We try:

$$T \cdot \tilde{b}_0 = \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{15}{4} \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_0(x)$$

This works the same with the other basis vectors $\tilde{b}_1 \ldots \tilde{b}_5$.

We can express the e_i basis in terms of the b_i basis:

$$e_0(x) = \tilde{b}_0(x)$$

$$e_1(x) = \tilde{b}_1(x)$$

$$e_2(x) = \tilde{b}_2(x) + \frac{1}{2}\tilde{b}_0(x)$$

$$e_3(x) = \tilde{b}_3(x) + \frac{3}{2}\tilde{b}_1(x)$$

$$e_4(x) = \tilde{b}_4(x) + 3\tilde{b}_2(x) + \frac{9}{8}\tilde{b}_0(x)$$

$$e_5(x) = \tilde{b}_5(x) + 5\tilde{b}_3(x) + \frac{15}{4}\tilde{b}_1(x)$$

We build the transformation matrix T^{-1}:

$$T^{-1} := \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{9}{8} & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{15}{4} \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We try:

$$T^{-1} \cdot e_0 = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{9}{8} & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{15}{4} \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = b_0(x)$$

This works the same with the other basis vectors $e_1 \dots e_5$.

We check whether $T \cdot T^{-1}$ gives the identity matrix:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{8} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{15}{4} \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{9}{8} & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{15}{4} \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Operator P

We examine the effect of the momentum operator P onto the $\tilde{b}_i$-basis.

The momentum operator P differentiates.

The structure of the orthogonal basis functions:

$$\tilde{b}_i = \frac{1}{2^i} e^{-\frac{x^2}{2}} H_i(x)$$

We differentiate according to the product rule:

$$\frac{d}{dx}\left(\frac{1}{2^i} e^{-\frac{x^2}{2}} H_i(x)\right) =$$

$$\left(e^{-\frac{x^2}{2}} \cdot (-2x) \cdot \frac{1}{2^i} \cdot H_i(x)\right) + \left(e^{-\frac{x^2}{2}} \cdot \frac{1}{2^i} \frac{d}{dx}\big(H_i(x)\big)\right) =$$

$$\frac{1}{2^i} e^{-\frac{x^2}{2}} \left(\big((-2x) \cdot H_i(x)\big) + \left(\frac{d}{dx}\big(H_i(x)\big)\right)\right)$$

We remember: Hermite polynomials are polynomials.

Differentiating a polynomial delivers another polynomial.

The sum of two polynomials gives a polynomial.

Multiplying a polynomial by x gives a new polynomial.

Polynomials can be broken down into basis functions.

Result: By applying the operator P we will not leave the vector space.

We try:

$$P\left(\tilde{b}_0(x)\right) = \frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right) = (-x)e^{-\frac{x^2}{2}} = -\tilde{b}_1(x)$$

$$P\left(\tilde{b}_1(x)\right) = \frac{d}{dx}\left(e^{-\frac{x^2}{2}} \cdot x\right) = -e^{-\frac{x^2}{2}}(x^2 - 1) = -e^{-\frac{x^2}{2}}\left(x^2 - \frac{1}{2} - \frac{1}{2}\right) = -\tilde{b}_2(x) + \frac{1}{2}\tilde{b}_0(x)$$

$$P\left(\tilde{b}_2(x)\right) = \frac{d}{dx}\left(e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right)\right) = \left(-x \cdot e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) + e^{-\frac{x^2}{2}} \cdot 2x\right) =$$

$$e^{-\frac{x^2}{2}}\left(-x^3 + \frac{5}{2}x\right) = -e^{-\frac{x^2}{2}}\left(x^3 - \frac{3}{2}x - x\right) = -e^{-\frac{x^2}{2}}\left(x^3 - \frac{3}{2}x\right) + e^{-\frac{x^2}{2}} \cdot x =$$

$$-\tilde{b}_3(x) + \tilde{b}_1(x)$$

$$P\left(\tilde{b}_3(x)\right) = \frac{d}{dx}\left(e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right)\right) = \left(-x \cdot e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) + e^{-\frac{x^2}{2}} \cdot \left(3x^2 - \frac{3}{2}\right)\right) =$$

$$e^{-\frac{x^2}{2}}\left(-x^4 + \frac{3}{2}x^2 + 3x^2 - \frac{3}{2}\right) = -e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 - \frac{3}{2}x^2 + \frac{3}{4} + \frac{3}{4}\right) =$$

$$-e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right) + e^{-\frac{x^2}{2}}\left(\frac{3}{2}x^2 - \frac{3}{4}\right) =$$

$$-\tilde{b}_4(x) + \frac{3}{2}\tilde{b}_2(x)$$

$$P\left(\tilde{b}_4(x)\right) = \frac{d}{dx}\left(e^{-\frac{x^2}{2}} \cdot \left(x^4 - 3x^2 + \frac{3}{4}\right)\right) = \left(-x \cdot e^{-\frac{x^2}{2}} \cdot \left(x^4 - 3x^2 + \frac{3}{4}\right) + e^{-\frac{x^2}{2}} \cdot (4x^3 - 6x)\right) =$$

$$\left(e^{-\frac{x^2}{2}} \cdot \left(-x^5 + 3x^3 - \frac{3}{4}x\right) + e^{-\frac{x^2}{2}} \cdot (4x^3 - 6x)\right) =$$

$$e^{-\frac{x^2}{2}}\left(-x^5 + 3x^3 - \frac{3}{4}x + 4x^3 - 6x\right) = e^{-\frac{x^2}{2}}\left(-x^5 + 7x^3 - \frac{27}{4}x\right) =$$

$$e^{-\frac{x^2}{2}}\left(-x^5 + 5x^3 + 2x^3 - \frac{15}{4}x - \frac{12}{4}x\right) = -e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right) + e^{-\frac{x^2}{2}}(2x^3 - 3x) =$$

$$-\tilde{b}_5(x) + 2\tilde{b}_3(x)$$

We try to find a scheme:

$$P\left(\tilde{b}_0(x)\right) = -\tilde{b}_1(x)$$

$$P\left(\tilde{b}_1(x)\right) = -\tilde{b}_2(x) + \frac{1}{2}\tilde{b}_0(x)$$

$$P\left(\tilde{b}_2(x)\right) = -\tilde{b}_3(x) + \tilde{b}_1(x)$$

$$P\left(b_3(x)\right) = -\tilde{b}_4(x) + \frac{3}{2}\tilde{b}_2(x)$$

$$P\left(b_4(x)\right) = -\tilde{b}_5(x) + 2\tilde{b}_3(x)$$

We get the matrix for the operator P:

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \dots & \dots \\ -1 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & -1 & 0 & \frac{3}{2} & 0 & \dots & \dots \\ 0 & 0 & -1 & 0 & 2 & \dots & \dots \\ 0 & 0 & 0 & -1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & -1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The structure of the matrix is simple. Below the diagonal we have entries -1. Above the diagonal we have the sequence:

$$\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \dots$$

This corresponds to the position scheme:

$$a_{ij} = \begin{cases} \frac{i}{2} & for\ j = i + 1 \\ -1 & for\ j = i - 1 \\ 0 & else \end{cases}$$

Operator X

We examine the effect of the position operator X onto the b_i-basis.

The position operator X multiplies by x.

$$X\left(\tilde{b}_0(x)\right) = x \cdot e^{-\frac{x^2}{2}} = b_1(x)$$

$$X\left(\tilde{b}_1(x)\right) = x \cdot e^{-\frac{x^2}{2}} \cdot x = e^{-\frac{x^2}{2}}\left(x^2 - \frac{1}{2} + \frac{1}{2}\right) = e^{-\frac{x^2}{2}}\left(x^2 - \frac{1}{2}\right) + \frac{1}{2}e^{-\frac{x^2}{2}} = \tilde{b}_2(x) + \frac{1}{2}\tilde{b}_0(x)$$

$$X\left(\tilde{b}_2(x)\right) = x \cdot e^{-\frac{x^2}{2}} \cdot \left(x^2 - \frac{1}{2}\right) = e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{1}{2}x\right) =$$

$$e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x + x\right) = e^{-\frac{x^2}{2}} \cdot \left(x^3 - \frac{3}{2}x\right) + e^{-\frac{x^2}{2}} \cdot x =$$

$$\tilde{b}_3(x) + \tilde{b}_1(x)$$

$$X\left(\tilde{b}_3(x)\right) = x \cdot \left(x^3 - \frac{3}{2}x\right) = e^{-\frac{x^2}{2}} \cdot \left(x^4 - \frac{3}{2}x^2\right) =$$

$$e^{-\frac{x^2}{2}} \cdot \left(x^4 - 3x^2 + \frac{3}{4} + \frac{3}{2}x^2 - \frac{3}{4}\right) = e^{-\frac{x^2}{2}} \cdot \left(x^4 - 3x^2 + \frac{3}{4}\right) + e^{-\frac{x^2}{2}} \cdot \left(\frac{3}{2}x^2 - \frac{3}{4}\right) =$$

$$b_4(x) + \frac{3}{2}\tilde{b}_3(x)$$

$$X\big(b_4(x)\big) = e^{-\frac{x^2}{2}}\left(x^4 - 3x^2 + \frac{3}{4}\right) = e^{-\frac{x^2}{2}}\left(x^5 - 3x^3 + \frac{3}{4}x\right) =$$

$$e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + 2x^3 + \frac{15}{4}x - \frac{12}{4}x\right) = e^{-\frac{x^2}{2}}\left(x^5 - 5x^3 + \frac{15}{4}x\right) + e^{-\frac{x^2}{2}}(2x^3 - 3x) =$$

$$\tilde{b}_5(x) + 2\tilde{b}_3(x)$$

We get the matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \dots & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & 2 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The structure of the matrix is simple. Below the diagonal we have entries 1. Above the diagonal we have the sequence:

$$\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \dots$$

This corresponds to the position scheme:

$$a_{ij} = \begin{cases} \frac{i}{2} & for\ j = i + 1 \\ 1 & for\ j = i - 1 \\ 0 & else \end{cases}$$

Summary

In the Hilbert space of square integrable modified polynomials we can express the process of differentiation by a matrix operator. This space is countably infinite, so is the matrix. The structure of the matrices is simple enough to express the coefficients by a position scheme.

The same holds for the process of multiplying a function by x.

For your convenience the normalizing factors:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot x\, dx = \int_{-\infty}^{\infty} e^{-x^2} \cdot x^3\, dx = \int_{-\infty}^{\infty} e^{-x^2} \cdot x^5\, dx = \dots = 0$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot x^2 \, dx = \frac{1}{2}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot x^4 \, dx = \frac{3}{4}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot x^6 \, dx = \frac{15}{8}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot x^8 \, dx = \frac{105}{16}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot x^{10} \, dx = \frac{945}{32}\sqrt{\pi}$$